# *Metric Space Inversions, Quasihyperbolic Distance, and Uniform Spaces*

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ABSTRACT. We define a notion of inversion valid in the general metric space setting. We establish several basic facts concerning inversions; e.g., they are quasimöbius homeomorphisms and quasihyperbolically bilipschitz. In a certain sense, inversion is dual to sphericalization. We demonstrate that both inversion and sphericalization preserve local quasiconvexity and annular quasiconvexity as well as uniformity.

CONTENTS

## 1. INTRODUCTION

The self-homeomorphism $x \mapsto x^* := x/|x|^2$ of $\mathbb{R}^n \setminus \{0\}$ (Euclidean $n$-dimensional space punctured at the origin) is often called inversion or reflection about the unit sphere centered at the origin. As is well-known, this is a Möbius transformation and moreover, a quasihyperbolic isometry. Also, a domain $\Omega \subset \mathbb{R}^n \setminus \{0\}$ is a uniform space if and only if its image $\Omega'$ is uniform. Furthermore, as a map $(\Omega, k) \to (\Omega', k')$ between the quasihyperbolizations of $\Omega$ and $\Omega'$, inversion is even 4-bilipschitz; see [18, Theorems 5.11, 5.12]. Observe that we can pull back Euclidean distance to obtain a new distance on $\mathbb{R}^n \setminus \{0\}$ via the formula

$$\|x - y\| := |x^* - y^*| = \left| \frac{x}{|x|^2} - \frac{y}{|y|^2} \right| = \frac{|x - y|}{|x|\,|y|}.$$

One can find this calculation, e.g., in [2, (3.1.5), p. 26].

In this article we extend the above ideas and results to general metric spaces $(X, d)$. Given a fixed base point $p \in X$, we define a distance function $d_p$ on $X_p := X \setminus \{p\}$ which satisfies

$$\frac{1}{4} i_p(x, y) \le d_p(x, y) \le i_p(x, y) := \frac{d(x, y)}{d(x, p)d(y, p)}.$$

Thus our notion of inversion is a direct generalization of inversion on punctured Euclidean space. The identity map $\mathrm{id} : (X_p, d) \to (X_p, d_p)$ is a quasimöbius homeomorphism; in particular, the topology induced by $d_p$ on $X_p$ agrees with its original subspace topology. In the metric $d_p$ the point $p \in X$ has been pushed out to infinity. See Section 3 for definitions, details and additional elementary properties.

In [4, Lemma 2.2, p. 87] Bonk and Kleiner define a metric on the one point compactification of an unbounded locally compact metric space. Their construction is a generalization of the deformation from the Euclidean distance on $\mathbb{R}^n$ to the chordal distance on its one point compactification. All of the properties of $d_p$

mentioned above also hold for their construction. Our notion of inversion is, in a certain sense, dual to the Bonk and Kleiner sphericalization. This duality is a consequence of two ideas: first, just as in the Euclidean setting, sphericalization can be realized as a special case of inversion (as explained at the end of Subsection 3.B); second, repeated inversions using the appropriate points produces a space which is bilipschitz equivalent to the original space.

In fact, we verify the following; see Propositions 3.3, 3.4, 3.5 for precise statements with explicit bilipschitz constants.

> *The natural identity maps associated with the following processes are bilipschitz:*
> - *inversion followed by inversion,*
> - *sphericalization followed by inversion,*
> - *inversion followed by sphericalization.*

These results are particularly useful when establishing various properties of inversion and sphericalization, especially with regards to subspaces. For example, we corroborate that inversions and sphericalizations are quasihyperbolically bilipschitz in the following sense; see Theorems 4.6 and 4.11.

> *If $\Omega$ is an open locally $c$-quasiconvex subspace of $X_p$, then both $(\Omega, k) \xrightarrow{\text{id}} (\Omega, k_p)$ and $(\Omega, k) \xrightarrow{\text{id}} (\Omega, \hat{k}_p)$ are bilipschitz.*

Moreover, the class of uniform subspaces is preserved; see Theorems 5.1 and 5.5.

> *A subspace is uniform if and only if its inversion is uniform. The same result is true regarding sphericalization.*

In Section 6 we introduce a notion called annular quasiconvexity and demonstrate that a space is quasiconvex and annular quasiconvex if and only if its inversion also enjoys these properties. The same result holds for sphericalization. See Theorems 6.4 and 6.5. Moreover, in the presence of an annular quasiconvex ambient space, we obtain improved quantitative information describing how uniformity constants change; see Theorems 6.7 and 6.8.

In [1] the first author and Balogh investigated a general notion for flattening and sphericalizing rectifiably connected spaces. Here our definitions are valid for all metric spaces. For annular quasiconvex spaces, our inversion is bilipschitz equivalent to the standard flattening. In Section 7, we explain this and briefly discuss a generalized notion of inversion.

The results in this paper are of great help when it is easier to establish a certain property for unbounded spaces rather than for bounded spaces, or vice versa. In particular, the second and third authors, together with Nageswari Shanmugalingam [10], make extensive use of our results to establish a characterization for uniform domains among Gromov hyperbolic domains in terms of whether or not a certain natural boundary correspondence is quasimöbius. In one direction, this result follows quickly for the bounded case; in the other (harder) direction, it is the unbounded case which is first dealt with.

We are indebted to Nages for suggesting the notion of annular quasiconvexity and for numerous helpful discussions.

This document is organized as follows: Section 2 contains basic definitions, terminology, and facts regarding metric spaces and quasihyperbolic distance. In Section 3 we define inversions and verify a number of useful properties. We investigate the quasihyperbolic geometry of inversions in Section 4. Our main theorems establishing the invariance of uniformity under inversion are presented in Section 5. In Section 6 we introduce the notion of annular quasiconvexity, demonstrate its invariance under inversion, and explain its connection with uniformity. We conclude by describing a generalized inversion in Section 7.

## 2. PRELIMINARIES

Here we set forth our basic notation, which is relatively standard, and provide fundamental definitions. We write $C = C(a, \dots)$ to indicate a constant $C$ which depends only on $a, \dots$ . Typically $a$, $b$, $c$, $C$, $K, \dots$ will be constants that depend on various parameters, and we try to make this as clear as possible, often giving explicit values. For real numbers we employ the notation

$$a \wedge b := \min\{a, b\} \quad \text{and} \quad a \vee b := \max\{a, b\}.$$

***2.A. General metric space information.*** In what follows $(X, d)$ will always denote a non-trivial metric space. For the record, this means that $X$ contains at least two points and that $d$ is a distance function on $X$. We often write the distance between $x$ and $y$ as $d(x, y) = |x - y|$ and $\mathrm{d}(x, A)$ is the distance from a point $x$ to a set $A$. The open ball (sphere) of radius $r$ centered at the point $x$ is $B(x; r) := \{y : |x - y| < r\}$ ($S(x; r) := \{y : |x - y| = r\}$). We write $A(x; r, R) := \{y : r \le |x - y| \le R\}$ for the closed *annular ring* centered at $x$ with inner and outer radii $r$ and $R$, respectively. A metric space is *proper* if it has the Heine-Borel property that every closed ball is compact. We let $\bar{X}$ denote the metric completion of a metric space $X$ and call $\partial X = \bar{X} \setminus X$ the metric boundary of $X$. We use $(\mathbb{R}^n, |\cdot|)$ to denote Euclidean $n$-space with Euclidean distance.

It is convenient to introduce the *one-point extension of* $X$ which is defined via

$$\hat{X} := \begin{cases} X & \text{when } X \text{ is bounded,} \\ X \cup \{\infty\} & \text{when } X \text{ is unbounded;} \end{cases}$$

a set $U \subset \hat{X}$ is open in $\hat{X}$ if and only if either $U$ is an open subset of $X$ or $\hat{X} \setminus U$ is a bounded closed subset of $X$. Thus when $X$ is a proper space, $\hat{X}$ is simply its one-point compactification. Given a subspace $Z \subset X$, we write $\hat{Z}$ and $\hat{\partial} Z$ to denote the closure and boundary of $Z$ in $\hat{X}$; e.g., $\hat{Z} = \bar{Z}$ when $Z$ is bounded and $\hat{Z} = \bar{Z} \cup \{\infty\}$ when $Z$ is unbounded.

A bijection $X \xrightarrow{f} Y$ between metric spaces is *$L$-bilipschitz* if $L \geq 1$ is some constant and

$$\forall\, x, y \in X: \quad L^{-1}|x-y| \leq |fx - fy| \leq L|x-y|.$$

We write $X \cong Y$ to mean that $X$ and $Y$ are bilipschitz equivalent. An *isometry* is a 1-bilipschitz homemorphism, and we write $X \equiv Y$ to mean that $X$ and $Y$ are isometric. More generally, $f : X \to Y$ is an *$(L, C)$-quasiisometry* if $L \geq 1$ and $C \geq 0$ are constants with

$$\forall\, x, y \in X: \quad L^{-1}|x-y| - C \leq |fx - fy| \leq L|x-y| + C$$

and

$$\forall\, z \in Y: \quad \text{there are some } x \in X \text{ with } |f(x) - z| \leq C.$$

An embedding $X \xrightarrow{f} Y$ is *$\vartheta$-quasimöbius* if $[0, \infty) \xrightarrow{\vartheta} [0, \infty)$ is a homeomorphism and for all quadruples $x$, $y$, $z$, $w$ of distinct points in $X$,

$$|x, y, z, w| := \frac{|x-y|\,|z-w|}{|x-z|\,|y-w|} \leq t \ \Rightarrow\ |fx, fy, fz, fw| \leq \vartheta(t).$$

These mappings were introduced and investigated by Väisälä in [15].

A *geodesic* in $X$ is the image $\varphi(I)$ of some isometric embedding $\mathbb{R} \supset I \xrightarrow{\varphi} X$ where $I$ is an interval; we use the phrases *segment*, *ray*, or *line* (respectively) to indicate that $I$ is bounded, semi-infinite, or all of $\mathbb{R}$. A metric space is *geodesic* if each pair of points can be joined by a geodesic segment. Given two points $x$ and $y$ on an *arc* $\alpha$ (the homemorphic image of an interval), we write $\alpha[x, y]$ to denote the subarc of $\alpha$ joining $x$ and $y$.

A metric space is *rectifiably connected* provided each pair of points can be joined by a rectifiable path. Such a space $(X, d)$ admits a natural *intrinsic* metric, its so-called *length distance* given by

$$l(x, y) := \inf\{\ell(\gamma) : \gamma \text{ a rectifiable path joining } x, y \text{ in } X\};$$

here $\ell(\gamma)$ denotes the length of $\gamma$. We call $(X, d)$ a *length* space provided $d(x, y) = l(x, y)$ for all points $x$, $y \in X$; it is also common to call such a $d$ an *intrinsic* distance function.

A path $\alpha$ with endpoints $x$, $y$ is *$c$-quasiconvex*, $c \geq 1$, if $\ell(\alpha) \leq c d(x, y)$. A metric space $(X, d)$ is *$c$-quasiconvex* if each pair of points can be joined by a $c$-quasiconvex path. We say that $(X, d)$ is *locally quasiconvex* provided it is connected and for each $x \in X$ there is a constant $c_x \geq 1$ and an open neighborhood $U_x$ of $x$ with the property that every pair of points in $U_x$ can be joined by a $c_x$-quasiconvex path. We say $X$ is *locally $c$-quasiconvex* if it is locally quasiconvex with $c_x = c$ for all $x \in X$.

By cutting out any loops, we can always replace a $c$-quasiconvex path with a $c$-quasiconvex arc. Fortunately, this intuitively clear idea has been made precise by Väisälä; see [17].

Since $|x - y| \leq l(x, y)$ for all $x$, $y$, the identity map $(X, l) \xrightarrow{\text{id}} (X, d)$ is Lipschitz continuous. It is important to know when this map will be a homeomorphism (cf. [3, Lemma A.4, p. 92]). Observe that if $X$ is locally quasiconvex, then $\text{id} : (X, l) \to (X, d)$ is a homeomorphism; $X$ is quasiconvex if and only if this map is bilipschitz.

***2.B. Quasihyperbolic distance.*** The quasihyperbolic distance in an incomplete locally compact rectifiably connected space $(X, d)$ is defined by

$$k(x, y) = k_X(x, y) := \inf_{\gamma} \ell_k(\gamma) := \inf_{\gamma} \int_{\gamma} \frac{|dz|}{\mathrm{d}(z, \partial X)}$$

where the infimum is taken over all rectifiable paths $\gamma$ which join $x$, $y$ in $X$. Here $X$ is incomplete, so $\partial X \neq \emptyset$, and $\mathrm{d}(z, \partial X)$ is the distance from $z \in X$ to the boundary $\partial X$ of $X$. We note that, as long as the identity map $(X, l) \to (X, d)$ is a homemorphism, $(X, k)$ will be complete, proper and geodesic; see [3, Proposition 2.8].

We call the geodesics in $(X, k)$ *quasihyperbolic geodesics*. We remind the reader of the following basic estimates for quasihyperbolic distance, first established by Gehring and Palka [8, Lemma 2.1], but see also [3, (2.3), (2.4)]:

$$\begin{aligned} k(x, y) &\geq \log\left(1 + \frac{l(x, y)}{\mathrm{d}(x, \partial X) \wedge \mathrm{d}(y, \partial X)}\right) \\ &\geq \log\left(1 + \frac{|x - y|}{\mathrm{d}(x, \partial X) \wedge \mathrm{d}(y, \partial X)}\right) \geq \left|\log \frac{\mathrm{d}(x, \partial X)}{\mathrm{d}(y, \partial X)}\right|. \end{aligned}$$

***Lemma 2.1.*** *Suppose $(X, d)$ is a locally $c$-quasiconvex, incomplete locally compact rectifiably connected space. Then each $x \in X$ has an open neighborhood $U$ with the property that*

$$\forall\, y \in U : \quad k(x, y) \leq \log \frac{\mathrm{d}(x, \partial X)}{\mathrm{d}(x, \partial X) - c|x - y|}.$$

*Proof.* Fix $x$ and let $U = U_x$ be the promised neighborhood in which points can be joined by $c$-quasiconvex arcs. Let $y \in U \cap B(x; \mathrm{d}(x, \partial X)/2c)$ and let

$\alpha$ be an arc joining $x$, $y$ with $L = \ell(\alpha) \le c|x-y|$. Then for each $z \in \alpha$, $\mathrm{d}(z,\partial X) \ge \mathrm{d}(x,\partial X) - \ell(\alpha[z,x])$ and thus

$$k(x,y) \le \ell_k(\alpha) \le \int_{s=0}^{L} \frac{ds}{\mathrm{d}(x,\partial X) - s}$$
$$= \log \frac{\mathrm{d}(x,\partial X)}{\mathrm{d}(x,\partial X) - L} \le \log \frac{\mathrm{d}(x,\partial X)}{\mathrm{d}(x,\partial X) - c|x-y|}. \qquad \square$$

## 3. Metric Space Inversions

Here we define what we mean by the inversion $\mathrm{Inv}_p(X)$ of a metric space $X$ with respect to a point $p \in X$. This comes with an associated distance function $d_p$, and we shall see that the 'identity' map $\hat{X} \setminus \{p\} \to \mathrm{Inv}_p(X)$ is $16t$-quasimöbius. Moreover, the composition of a suitable pair of inversions gives a bilipschitz map, which is analagous to the fact that Euclidean space inversions have order 2.

In a certain sense, our definition is dual to a similar construction of Bonk and Kleiner; see [4, p. 87], Subsection 3.B, and Propositions 3.4, 3.5.

***3.A. Definitions and basic properties.*** Let $(X,d)$ be a metric space and fix a base point $p \in X$. Consider the quantity

$$i_p(x,y) := \frac{d(x,y)}{d(x,p)d(y,p)},$$

which is defined for $x$, $y \in X_p := X \setminus \{p\}$; sometimes this is a distance function, but in general it may not satisfy the triangle inequality. For instance, if we supply $X = \mathbb{R}^2$ with the $l^1$ metric $d(x,y) = |x_1 - y_1| + |x_2 - y_2|$, where $x = (x_1,x_2)$ and $y = (y_1,y_2)$, then taking $p = (0,0)$, $u = (1,0)$, $v = (0,1)$, and $w = (1,1)$, we find that $i_p(u,v) = 2$ but $i_p(u,w) + i_p(w,v) = 1$.

Fortunately there is a standard technique which forces the triangle inequality: we define

$$d_p(x,y) := \inf\Big\{ \sum_{i=1}^{k} i_p(x_i,x_{i-1}) : x = x_0,\dots,x_k = y \in X_p \Big\}.$$

We shall see below in Lemma 3.1 that for all $x$, $y \in X_p$,

$$\frac{1}{4} i_p(x,y) \le d_p(x,y) \le i_p(x,y) \le \frac{1}{d(x,p)} + \frac{1}{d(y,p)}. \tag{3.1}$$

In particular, we deduce that $d_p$ is an honest distance function on $X_p$. Moreover, we see that when our original space $(X,d)$ is unbounded, there is a unique point $p'$ in the completion of $(X_p,d_p)$ which corresponds to the point $\infty$ in $\hat{X}$. (Indeed, any unbounded sequence in $(X_p, d)$ is a Cauchy sequence in $(X_p, d_p)$, and any two such sequences are equivalent.) Because of this phenomenon, we define

$$(\mathrm{Inv}_p(X), d_p) := (\hat{X}_p, d_p) = (\hat{X} \setminus \{p\}, d_p)$$

and we call $(\mathrm{Inv}_p(X), d_p)$ the *inversion* of $(X, d)$ with respect to the base point $p$. For example, with this definition, $\mathrm{Inv}_p(X)$ will be complete (or proper) whenever $X$ is complete (or proper). Notice that other properties, such as connectedness and local compactness, are not necessarily preserved; e.g., connectedness is reversed for the subsets $[-1, 1]$ and $\mathbb{R} \setminus (0, 1)$ of the Euclidean line when they are inverted with respect to the origin.

The distance function $d_p$ on $X_p$ extends in the usual way to $\hat{X}_p$. Alternatively, when $X$ is unbounded, we can define

$$\forall\, x \in X_p : \quad i_p(x, p') = i_p(x, \infty) := \frac{1}{d(x, p)},$$

and then check that the definition of $d_p(x, y)$ using auxiliary points in $X_p$ is the same as using points in $\hat{X}_p$.

The metric quantities in $\mathrm{Inv}_p(X)$ are denoted by using a subscript $p$. For example: $B_p(x; r)$ and $A_p(x; r, R)$ are a $d_p$-ball and a $d_p$-annular ring centered at $x$, respectively; $\mathrm{d}_p(x, A)$, $\mathrm{diam}_p(A)$, and $\ell_p(\gamma)$ are the $d_p$-distance from a point to a set, the $d_p$-diameter of a set, and the $d_p$-length of a path, respectively.

As an elementary example, the reader can confirm from our definitions that $\mathrm{Inv}_0(\mathbb{R}^n) \equiv \mathbb{R}^n$, or more precisely,

$$(\mathrm{Inv}_0(\mathbb{R}^n), |\cdot|_0) = (\hat{\mathbb{R}}^n_0, \|\cdot\|) \equiv (\mathbb{R}^n, |\cdot|),$$

where—as in Section 1—the isometry is provided by the standard inversion $x \mapsto x^* = x/|x|^2$ (with $\infty \in \hat{\mathbb{R}}^n_0$, i.e., $0' \in \mathrm{Inv}_0(\mathbb{R}^n)$, corresponding to 0) and $|x - y|_0 = \|x - y\| = |x^* - y^*|$.

Frink [5] employed a similar chaining argument to get a metric comparable to an original quasimetric which was also smaller by at most a factor of 4. Mineyev [13] has used $i_p(x, y)$ to define a new metric on the one point complement of the Gromov boundary of hyperbolic complexes. Ibragimov [11] has studied the geometry of Euclidean domains $D$ using the so-called *Cassinian metric* $c_D(x, y) = \sup_{p \in \partial D} i_p(x, y)$.

Here are some elementary properties of inversion. The proofs of parts (a) and (b), which are essentially the same as in [4, Lemma 2.2], are included for the reader's convenience; (b) generalizes the fact that inversions in Euclidean spaces are Möbius transformations.

***Lemma 3.1.*** *Let $(X, d)$ be a metric space and fix a base point $p \in X$.*

(a) *The inequalities in (3.1) hold for all points $x$, $y \in \mathrm{Inv}_p(X)$. In particular, $d_p$ is a distance function on $\mathrm{Inv}_p(X)$.*

(b) *The identity map* $(X_p, d) \xrightarrow{\text{id}} (X_p, d_p)$ *is a* $16t$*-quasimöbius homemorphism.*
(c) $\mathrm{Inv}_p(X)$ *is bounded if and only if* $p$ *is an isolated point in* $(X, d)$ *in which case*

$$\frac{\operatorname{diam} X_p}{\mathrm{d}(p, X_p) + \operatorname{diam} X_p} \frac{1}{8\,\mathrm{d}(p, X_p)} \le \operatorname{diam}_p \mathrm{Inv}_p(X) \le \frac{2}{\mathrm{d}(p, X_p)}.$$

*Proof.* (a) It suffices to verify the inequalities in (3.1) for $x$, $y \in X_p$, for if $X$ is unbounded and one of these points happens to be $p'$, then we simply look at the appropriate limit. The right hand inequalities there follow directly from the definitions of $d_p$ and $i_p$. In order to prove the left most inequality, we define $h_p : X_p \to [0, \infty)$ by $h_p(z) = 1/d(z, p)$; then $i_p(x, y) = h_p(x)h_p(y)d(x, y)$. We assume $d(x, p) \le d(y, p)$, so $h_p(x) \ge h_p(y)$.

Let $x_0, \ldots, x_k$ be an arbitrary sequence of points in $X_p$ with $x_0 = x$ and $x_k = y$. We consider two cases. If $h_p(x_i) \ge \frac{1}{2}h_p(x)$ for all $i$, then the triangle inequality applied to $d$ gives

$$\begin{aligned}\sum_{i=1}^{k} i_p(x_i, x_{i-1}) &\ge \frac{1}{4}h_p(x)^2 \sum_{i}^{k} d(x_i, x_{i-1}) \\ &\ge \frac{1}{4}d(x, y)h_p(x)h_p(y) = \frac{1}{4}i_p(x, y).\end{aligned}$$

Suppose instead that there exists some $j \in \{0, \ldots, k\}$ such that $h_p(x_j) < \frac{1}{2}h_p(x)$. Note that from the definitions, $|h_p(u) - h_p(v)| \le i_p(u, v)$ for $u$, $v \in X_p$. Since $d(x, p) \le d(y, p)$, $d(x, y) \le 2d(y, p)$, so $i_p(x, y) \le 2h_p(x)$. Thus again we arrive at

$$\sum_{i=1}^{k} i_p(x_i, x_{i-1}) \ge \sum_{i=1}^{k} |h_p(x_i) - h_p(x_{i-1})| \ge \frac{1}{2}h_p(x) \ge \frac{1}{4}i_p(x, y).$$

(b) This follows from (3.1) by observing that whenever $x, y, z, w$ is a quadruple of distinct points in $X_p$,

$$\frac{d_p(x, y)d_p(z, w)}{d_p(x, z)d_p(y, w)} \le 16\frac{i_p(x, y)i_p(z, w)}{i_p(x, z)i_p(y, w)} = 16\frac{d(x, y)d(z, w)}{d(x, z)d(y, w)}.$$

(c) It is straightforward to see that $\mathrm{Inv}_p(X)$ is unbounded precisely when $p$ is a non-isolated point in $X$. Finally, suppose $\delta = \mathrm{d}(p, X_p) > 0$. The estimates in (3.1) immediately give $\operatorname{diam}_p \mathrm{Inv}_p(X) \le 2/\delta$. For the lower estimate, we may assume that $\operatorname{diam} X_p > 0$. Let $0 < \varepsilon < \delta \wedge \operatorname{diam}(X_p)$ and pick $a, b \in \mathrm{Inv}_p(X)$ with $d(a, p) \le \delta + \varepsilon$ and $d(a, b) \ge (\operatorname{diam} X_p - \varepsilon)/2$. Then $d(b, p) \le \delta +$

$\operatorname{diam} X_p + \varepsilon$, so

$$\operatorname{diam}_p \operatorname{Inv}_p(X) \geq d_p(a,b) \geq \frac{1}{4}\frac{d(a,b)}{d(a,p)d(b,p)} \geq \frac{\operatorname{diam} X_p - \varepsilon}{8(\delta+\varepsilon)(\delta + \operatorname{diam} X_p + \varepsilon)}.$$

Letting $\varepsilon \to 0$ we deduce the asserted lower bound. □

***3.B. Sphericalization.*** As mentioned already, our definition of $\operatorname{Inv}_p(X)$ mimics a construction of Bonk and Kleiner [4, Lemma 2.2, p. 87]. We briefly recall their work. Let $(X,d)$ be any metric space, fix a base point $p \in X$, and consider

$$s_p(x,y) := \frac{d(x,y)}{[1+d(x,p)][1+d(y,p)]}$$

which is defined for $x$, $y \in X$. Sometimes this is a distance function, but in general it may not satisfy the triangle inequality, so we define

$$\hat{d}_p(x,y) := \inf\Big\{\sum_{i=1}^{k} s_p(x_i,x_{i-1}) : x = x_0,\dots,x_k = y \in X\Big\}.$$

Then for all $x$, $y \in X$,

$$\text{(3.2)} \qquad \frac{1}{4}s_p(x,y) \leq \hat{d}_p(x,y) \leq s_p(x,y) \leq \frac{1}{1+d(x,p)} + \frac{1}{1+d(y,p)}.$$

In particular, $\hat{d}_p$ is a distance function on $X$ and the map $(X,d) \xrightarrow{\text{id}} (X,\hat{d}_p)$ is a $16t$-quasimöbius homemorphism. Moreover, we see that when our original space $X$ is unbounded, there is a unique point $\hat{p}$ in the completion of $(X,\hat{d}_p)$ which corresponds to the point $\infty$ in $\hat{X}$. We define the *sphericalization* of $(X,d)$ with respect to the base point $p$ by

$$(\operatorname{Sph}_p(X), \hat{d}_p) := (\hat{X}, \hat{d}_p).$$

The distance function $\hat{d}_p$ on $X$ extends in the usual way to $\hat{X}$. Alternatively, when $X$ is unbounded, we define

$$\forall\, x \in X_p : \quad s_p(x,\hat{p}) = s_p(x,\infty) := \frac{1}{1+d(x,p)},$$

and then check that the definition of $\hat{d}_p(x,y)$ using auxiliary points in $X$ is the same as using points in $\hat{X}$. Note too that the metric topology induced by $\hat{d}_p$ on $\hat{X}$ is the one-point extension topology; that is, the identity map $\operatorname{Sph}_p(X) \to \hat{X}$ is a homemorphism.

The metric quantities in $\mathrm{Sph}_p(X)$ are denoted by using both a hat and a subscript $p$. For example: $\hat{B}_p(x;r)$ and $\hat{A}_p(x;r,R)$ are a $\hat{d}_p$-ball and a $\hat{d}_p$-annular ring centered at $x$, respectively; $\hat{\mathrm{d}}_p(x,A)$, $\widehat{\mathrm{diam}}_p(A)$, $\hat{\ell}_p(\gamma)$ are the $\hat{d}_p$-distance from a point to a set, the $\hat{d}_p$-diameter of a set and the $\hat{d}_p$-length of a path, respectively.

An elementary example is provided by

$$(\mathrm{Sph}_0(\mathbb{R}^n), |\hat{\cdot}|_0) = (\hat{\mathbb{R}}^n, |\hat{\cdot}|_0) \cong (\mathbb{S}^n, |\cdot|) \subset (\mathbb{R}^{n+1}, |\cdot|);$$

we check that stereographic projection $(\hat{\mathbb{R}}^n, |\hat{\cdot}|_0) \to (\mathbb{S}^n, |\cdot|)$ is 16-bilipschitz by using the estimate $\sqrt{1+t^2} \le 1+t \le \sqrt{2}\sqrt{1+t^2}$ which is valid for all $t \ge 0$.

The inequalities in (3.2) continue to hold for all points in $\hat{X}$. Moreover,

$$\widehat{\mathrm{diam}}_p \, \mathrm{Sph}_p(X) \le 1$$

since $d(x,p) + d(y,p) \le [1+d(x,p)][1+d(y,p)]$), and

$$\widehat{\mathrm{diam}}_p \, \mathrm{Sph}_p(X) \ge \begin{cases} \hat{d}_p(p,\hat{p}) \ge \dfrac{1}{4} & \text{when } X \text{ is unbounded,} \\ \dfrac{1}{4} \dfrac{\mathrm{diam}\, X}{2 + \mathrm{diam}\, X} & \text{when } X \text{ is bounded.} \end{cases}$$

The lower bound for bounded $X$ follows by estimating $\hat{d}_p(x,p)$ where $d(x,p) > (\mathrm{diam}(X) - \varepsilon)/2$, and then letting $\varepsilon \to 0$.

There is a more significant relation between inversion and sphericalization: the latter is a special case of the former. Fix $p \in X$. Put $X^q := X \sqcup \{q\}$, the disjoint union of $X$ and some point $q$, and define $d^{p,q} : X^q \times X^q \to \mathbb{R}$ by

$$d^{p,q}(x,y) := d^{p,q}(y,x) := \begin{cases} 0 & \text{if } x = q = y, \\ d(x,y) & \text{if } x \ne q \ne y, \\ d(x,p) + 1 & \text{if } x \ne q = y. \end{cases}$$

Then $(X^q, d^{p,q})$ is a metric space and $(\mathrm{Inv}_q(X^q), (d^{p,q})_q)$ is (isometric to) $(\mathrm{Sph}_p(X), \hat{d}_p)$.

Note that this idea of viewing sphericalization as a special case of inversion is a direct analog of the Euclidean setting where stereographic projection from $\hat{\mathbb{R}}^n$ to $\mathbb{S}^n$ can be viewed as inversion about the sphere $S(e_{n+1}; \sqrt{2}) \subset \mathbb{R}^{n+1}$; see, e.g., [2, Ex.8 on p. 27 or Subsection 3.4].

If we wish to restrict attention to quasiconvex spaces, it is useful to define sphericalization as a special case of inversion by adding a line segment to $X$ rather than a single point $q$. Let $X^{(0,1]} := X \sqcup (0,1]$ and consider the distance $d^{p,(0,1]}$

which restricts to $d$ on $X$, restricts to the Euclidean metric on $(0,1]$, and satisfies $d^{p,(0,1]}(x,t) = d(x,p)+t$ for points $x \in X$ and $t \in (0,1]$. As before, if we let $q$ denote the point 1 in $X^{(0,1]}$, then $(\mathrm{Inv}_q(X^{(0,1]}) \setminus (0,1), (d^{p,(0,1]})_q)$ is (isometric to) $(\mathrm{Sph}_p(X), \hat{d}_p)$.

***3.C. Elementary mapping properties.*** Here we examine the effects of inversion followed by another inversion or sphericalization, and sphericalization followed by inversion. We demonstrate that the associated natural identity maps are bilipschitz.

Before embarking on this investigation, we mention that inversions and sphericalizations are local quasidilations in the following sense. For $p \in X$ and $0 < r \le R$ we have

$$\forall\, x,y \in A(p;r,R): \qquad \begin{aligned} \frac{d(x,y)}{4R^2} &\le d_p(x,y) \le \frac{d(x,y)}{r^2}, \\ \frac{d(x,y)}{4(1+R)^2} &\le \hat{d}_p(x,y) \le \frac{d(x,y)}{(1+r)^2}. \end{aligned}$$

It is possible to use these inequalities to show that inversions and sphericalizations both preserve local quasiconvexity. See Proposition 4.2 for an alternative argument.

We record one more elementary observation.

***Lemma 3.2.*** *Suppose* $X \xrightarrow{f} Y$ *is* $K$*-bilipschitz. Then the induced maps* $\mathrm{Inv}_p(X) \to \mathrm{Inv}_{f(p)}(Y)$ *and* $\mathrm{Sph}_p(X) \to \mathrm{Sph}_{f(p)}(Y)$ *are* $4K^3$*-bilipschitz.*

Recall that when $X$ is unbounded and $p \in X$, we denote by $p'$ and $\hat{p}$ the points in $\mathrm{Inv}_p(X)$ and $\mathrm{Sph}_p(X)$ (respectively) which correspond to $\infty$ in $\hat{X}$. In this setting we have

$$\forall\, x \in X_p: \quad \frac{1}{4}\frac{1}{d(x,p)} \le d_p(x,p') \le \frac{1}{d(x,p)}, \tag{3.3}$$

$$\forall\, x \in X: \quad \frac{1}{4}\frac{1}{1+d(x,p)} \le \hat{d}_p(x,\hat{p}) \le \frac{1}{1+d(x,p)}. \tag{3.4}$$

Now we turn our attention to repeated inversions. Suppose $X$ is unbounded. Let $p \in X$ and $(Y,e) = (\mathrm{Inv}_p(X), d_p)$. Since $p'$ is a non-isolated point of $Y$, $\mathrm{Inv}_{p'}(Y) = \hat{Y}_{p'}$ is unbounded. When $Y$ is bounded, $\hat{Y} = Y = \hat{X}_p$, so $\hat{Y}_{p'} = Y_{p'} = X_p$ and there is a natural identity map

$$X_p \xrightarrow{\mathrm{id}} \mathrm{Inv}_{p'}(\mathrm{Inv}_p X) \quad \text{when } X \text{ is unbounded and } p \text{ is isolated.}$$

On the other hand, suppose $Y$ is unbounded (i.e., $p$ is a non-isolated point in $X$). Then $p \in X$ corresponds to $p'' \in \mathrm{Inv}_{p'}(Y)$, the unique point in the completion of $(Y_{p'}, e_{p'})$ which corresponds to $\infty$ in $\hat{Y}$. Thus, $\hat{Y} = \hat{X}$ and $\hat{Y}_{p'} = X$, so there is a natural identity map

$$X \xrightarrow{\text{id}} \mathrm{Inv}_{p'}(\mathrm{Inv}_p X) \quad \text{when } X \text{ is unbounded and } p \text{ is non-isolated}$$

(again, where $p \in X$ corresponds to $p''$).

With the above conventions in place, we now establish an analogue of the fact that Euclidean space inversions have order two.

***Proposition 3.3.*** *Let $(X, d)$ be an unbounded metric space. Fix $p \in X$ and let $p' \in Y = \mathrm{Inv}_p(X)$ correspond to $\infty \in \hat{X}$. Let $d' = (d_p)_{p'}$ denote the distance on $\mathrm{Inv}_{p'}(Y) = \mathrm{Inv}_{p'}(\mathrm{Inv}_p X)$.*

(a) *If $p$ is a non-isolated point, then the identity map*

$$(X, d) \xrightarrow{\text{id}} (X, d') = (\mathrm{Inv}_{p'}(Y), d')$$

*(where $p \mapsto p''$) is 16-bilipschitz.*

(b) *If $p$ is isolated, then the identity map $(X_p, d) \xrightarrow{\text{id}} (X_p, d') = (\mathrm{Inv}_{p'}(Y), d')$ is 16-bilipschitz.*

*Proof.* We establish (a); the proof for (b) is similar and simpler. Recalling (3.3), and noting that there are similar inequalities with $p$, $p'$, $d$, $d_p$ replaced by $p'$, $p''$, $d_p$, $d'$ respectively, we see that for $x \in Y_{p'} = X_p$,

$$\frac{d(x,p)}{4} \le \frac{1}{4d_p(x,p')} \le d'(x,p'') \le \frac{1}{d_p(x,p')} \le 4d(x,p).$$

Next, for $x$, $y \in X_p$ we have

$$\begin{aligned} d'(x,y) &\le \frac{d_p(x,y)}{d_p(x,p')d_p(y,p')} \\ &\le \frac{d(x,y)}{d(x,p)d(y,p)} \cdot 4d(x,p) \cdot 4d(y,p) = 16d(x,y) \end{aligned}$$

and similarly $d'(x,y) \ge d(x,y)/16$. □

Our next two results illustrate how inversion is in a certain sense dual to sphericalization. First we examine the effect of sphericalization followed by inversion. Suppose $X$ is unbounded, $p \in X$, $Y = \mathrm{Sph}_p(X) = \hat{X}$. Then $\hat{p} \in Y$ is a non-isolated point, so $\mathrm{Inv}_{\hat{p}}(Y)$ is also unbounded and as sets, $\mathrm{Inv}_{\hat{p}}(Y) = \hat{Y}_{\hat{p}} = X$. Thus there is a natural identity map

$$X \xrightarrow{\text{id}} \mathrm{Inv}_{\hat{p}}(\mathrm{Sph}_p X) \quad \text{when } X \text{ is unbounded.}$$

***Proposition 3.4.*** *Let $(X,d)$ be an unbounded metric space and fix $p \in X$. Put $Y = \mathrm{Sph}_p(X)$ and let $d' = (\hat{d}_p)_{\hat{p}}$ denote the distance on $\mathrm{Inv}_{\hat{p}}(Y) = \mathrm{Inv}_{\hat{p}}(\mathrm{Sph}_p X)$. Then the identity map $(X,d) \xrightarrow{\text{id}} (\mathrm{Inv}_{\hat{p}}(Y), d') = (X, d')$ is 16-bilipschitz.*

*Proof.* Let $x, y \in X$. Then

$$
\begin{aligned}
d'(x,y) &\le \frac{\hat{d}_p(x,y)}{\hat{d}_p(x,\hat{p})\hat{d}_p(y,\hat{p})} \\
&\le 16[1 + d(x,p)]\hat{d}_p(x,y)[1 + d(y,p)] \le 16 d(x,y)
\end{aligned}
$$

and similarly,

$$
\begin{aligned}
d'(x,y) &\ge \frac{\hat{d}_p(x,y)}{4\hat{d}_p(x,\hat{p})\hat{d}_p(y,\hat{p})} \\
&\ge \frac{1}{4}[1 + d(x,p)]\hat{d}_p(x,y)[1 + d(y,p)] \ge \frac{1}{16} d(x,y).
\end{aligned}
$$
□

Since sphericalization is a special case of inversion, the previous lemma also follows from Proposition 3.3: first add an isolated point $q$ to $X$ to get $X^q$, then invert with respect to $q$ to get $\mathrm{Inv}_q(X^q) \equiv \mathrm{Sph}_p(X)$, and then invert with respect to $q' = \hat{p}$ (the point $\infty$ in $\hat{X}$); thus $\mathrm{Inv}_{\hat{p}}(\mathrm{Sph}_p X)$ is just the iterated inversion $\mathrm{Inv}_{q'}(\mathrm{Inv}_q X^q)$.

Next we examine the effect of inversion followed by sphericalization. Recall that for an unbounded space, any sphericalization has diameter in $[\frac{1}{4}, 1]$. As above, let $p \in X$ and suppose $Y = \mathrm{Inv}_p(X) = \hat{X}$ is unbounded. Given any $q \in Y$, we see that $p$ corresponds to $\hat{q}$, the unique point in the completion of $(Y, \hat{e}_q)$ which corresponds to $\infty$ in $\hat{Y}$. Thus $\mathrm{Sph}_q(Y) = \hat{Y} = \hat{X}$ ($= X$ if $X$ is bounded) and there is a natural identity map

$$\hat{X} \xrightarrow{\text{id}} \mathrm{Sph}_q(\mathrm{Inv}_p X) \quad \text{when } p \text{ is non-isolated}$$

where $p \in X$ corresponds to $\hat{q}$ (and $\infty$ corresponds to $p'$ if $X$ is unbounded).

***Proposition 3.5.*** *Let $(X,d)$ be a metric space with $\mathrm{diam}(X) = 1$. Suppose $p$ is a non-isolated point in $X$ and there exists a point $q \in X$ with $d(p,q) \ge \frac{1}{2}$. Put $(Y,e) = (\mathrm{Inv}_p(X), d_p)$. Then*

$$(X,d) \xrightarrow{\text{id}} (\mathrm{Sph}_q(Y), \hat{e}_q) = (X, \hat{e}_q) \quad (\text{where } p \mapsto \hat{q})$$

*is 256-bilipschitz.*

*Proof.* Here $\hat{e}_q$ is obtained from

$$t_q(x,y) := \frac{e(x,y)}{[1+e(x,q)][1+e(y,q)]}$$

in the same manner that $\hat{d}_p$ is obtained from $s_p$. First we check that

$$\forall\, x \in X_p : \quad \frac{1}{4}d(x,p) \le t_q(x,\hat{q}) \le 4d(x,p).$$

Let $x \in X_p$. We have

$$\begin{aligned} t_q(x,\hat{q}) &= \frac{1}{1+d_p(x,q)} \le \frac{4d(x,p)d(q,p)}{4d(x,p)d(q,p)+d(x,q)} \\ &\le \frac{4d(x,p)d(q,p)}{2d(x,p)+d(x,q)} \le 4d(x,p) \end{aligned}$$

and

$$\begin{aligned} t_q(x,\hat{q}) &= \frac{1}{1+d_p(x,q)} \ge \frac{d(x,p)d(q,p)}{d(x,p)d(q,p)+d(x,q)} \\ &\ge \frac{1}{2}\frac{d(x,p)}{d(x,p)+d(x,q)} \ge \frac{1}{4}d(x,p). \end{aligned}$$

Since $t_q(x,\hat{q})/4 \le \hat{e}_q(x,\hat{q}) \le t_q(x,\hat{q})$, the above yields

$$\forall\, x \in X_p : \quad \frac{1}{16}d(x,p) \le \hat{e}_q(x,\hat{q}) \le 4d(x,p).$$

Finally, we consider points $x$, $y \in X_p$. Notice that

$$\begin{aligned} t_q(x,y) &= \frac{d_p(x,y)}{[1+d_p(x,q)][1+d_p(y,q)]} \\ &= t_q(x,\hat{q})d_p(x,y)t_q(y,\hat{q}). \end{aligned}$$

The estimates from above produce $d(x,y)/64 \le t_q(x,y) \le 16d(x,y)$ from which it follows that $d(x,y)/256 \le \hat{e}_q(x,y) \le 16d(x,y)$. ◻

In general, when $X$ is any bounded metric space and $p \in X$ is non-isolated, the map

$$(X,d) \xrightarrow{\text{id}} (\operatorname{Sph}_q(\operatorname{Inv}_p X), \hat{e}_q) \quad \text{where } e = d_p$$

is still bilipschitz, but the distortion constant will always depend on $\operatorname{diam}(X)$ (because $\widehat{\operatorname{diam}}\,\operatorname{Sph}_q(Y) \le 1$) and may also depend on $d(p,q)$. For an example to see the latter dependence, invert $X = [0,1]$ with respect to $p = 0$ and take $q$ close to $p$.

***3.D. Subspaces and notation.*** For later use, here we set forth some notational conventions. Let $\Omega$ be an open subspace of $(X,d)$. When $\partial\Omega \neq \emptyset$, we let $\delta(x) = \delta_\Omega(x) := \mathrm{d}(x,\partial\Omega)$ denote the distance (in $(X,d)$) from a point $x$ to $\partial\Omega$.

If $\Omega \subset X_p = X \setminus \{p\}$ for some fixed base point $p \in X$, we can view $\Omega$ as a subspace of $(\operatorname{Inv}_p(X), d_p)$ or $(\operatorname{Sph}_p(X), \hat{d}_p)$. To indicate this we write $I_p(\Omega) := (\Omega, d_p)$ or $S_p(\Omega) := (\Omega, \hat{d}_p)$. Then

$$\partial_p\Omega,\ \delta_p(x) := \mathrm{d}_p(x,\partial_p\Omega) \quad \text{and} \quad \hat{\partial}_p\Omega,\ \hat{\delta}_p(x) := \hat{\mathrm{d}}_p(x,\hat{\partial}_p\Omega)$$

denote the boundary of $\Omega$ and the distance to it in $(\operatorname{Inv}_p(X), d_p)$ and in $(\operatorname{Sph}_p(X), \hat{d}_p)$, respectively. Notice that $\partial\Omega_p = \partial\Omega \setminus \{p\}$ when $\Omega$ is bounded, whereas if $\Omega$ is unbounded, $\partial\Omega_p = (\partial\Omega \setminus \{p\}) \cup \{p'\}$.

We conclude this (sub)section with estimates for various distances to boundaries.

***Lemma 3.6.*** *Let $\Omega \subset X_p = X \setminus \{p\}$ (for some fixed base point $p \in X$) be an open subspace of $(X,d)$. Then for all $x \in \Omega$:*

(a) $$\delta_p(x) \ge \frac{1}{8}\left(\frac{1}{d(x,p)} \wedge \frac{\delta(x)}{d(x,p)^2}\right),$$

(b) $$\delta(x) \ge \frac{1}{2}\big(d(x,p) \wedge \delta_p(x) d(x,p)^2\big),$$

(c) $$\hat{\delta}_p(x) \ge \frac{1}{16}\left(\frac{1}{1+d(x,p)} \wedge \frac{\delta(x)}{[1+d(x,p)]^2}\right),$$

(d) $$\delta(x) \ge \frac{1}{4}\big([1+d(x,p)] \wedge (\hat{\delta}_p(x)[1+d(x,p)]^2)\big).$$

*Proof.* For each part, we would like to choose a closest boundary point. Since these may not exist, we must use an approximation argument. We do this explicitly only for part (a). If any of these distances to the boundary are infinite, there is nothing to prove, so we assume they are all finite. Parts (c) and (d) are not needed in this paper but are included as they may be of use elsewhere; we leave their proofs to the reader.

(a) Let $\varepsilon > 0$. Select $a \in \partial_p\Omega$ with $\delta_p(x) \ge d_p(x,a) - \varepsilon$. If $\Omega$ is unbounded and $a = p'$, then $\delta_p(x) + \varepsilon \ge d_p(x,p') \ge 1/(4d(x,p))$. Assume either $\Omega$ is bounded or $a \neq p'$. If $d(a,p) \leq 2d(x,p)$, then

$$i_p(x,a) \geq \frac{1}{2}\frac{d(x,a)}{d(x,p)^2} \geq \frac{1}{2}\frac{\delta(x)}{d(x,p)^2}$$

so

$$\delta_p(x) + \varepsilon \geq \frac{1}{8}\frac{\delta(x)}{d(x,p)^2}.$$

If $d(a,p) \geq 2\,d(x,p)$, then $d(x,a) \geq d(a,p) - d(x,p) \geq \frac{1}{2}d(a,p)$, so

$$i_p(x,a) \geq \frac{1}{2}\frac{1}{d(x,p)}$$

and, again,

$$\delta_p(x) + \varepsilon \geq \frac{1}{8}\frac{1}{d(x,p)}.$$

Letting $\varepsilon \to 0$ yields the asserted conclusion.

(b) Suppose $a \in \partial\Omega$ gives $\delta(x) = d(x,a)$. If $d(a,p) \leq d(x,p)/2$, then $\delta(x) \geq d(x,p) - d(a,p) \geq d(x,p)/2$. Assume $d(a,p) \geq d(x,p)/2$. Then $a \neq p$, so $a \in \partial_p\Omega$ and therefore

$$\delta_p(x) \leq d_p(x,a) \leq i_p(x,a) \leq 2\frac{d(x,a)}{d(x,p)^2} = 2\frac{\delta(x)}{d(x,p)^2}. \qquad \square$$

## 4. Inversions and Quasihyperbolic Distance

Our main goal here is to establish Theorem 4.6 which asserts that inversions induce bilipschitz maps when viewed in the associated quasihyperbolic metrics. As a consequence we obtain Theorem 4.11 which says that the same holds for sphericalizations.

***4.A. Linear distortion.*** Suppose $X \xrightarrow{f} Y$ is a map between metric spaces and let $x \in X$ be a non-isolated point of $X$. We write

$$L(x,f) := \limsup_{y\to x} \frac{d(f(x),f(y))}{d(x,y)},$$

$$l(x,f) := \liminf_{y\to x} \frac{d(f(x),f(y))}{d(x,y)}$$

to denote the *maximal* and *minimal stretching* of $f$ at $x$ respectively.

***Proposition 4.1.*** *Given $p \in X$, the identity map $(X_p, d) \xrightarrow{\text{id}} (X_p, d_p)$ satisfies*

$$\text{for each non-isolated point } x \in X_p : \quad l(x, \text{id}) = L(x, \text{id}) = d(x, p)^{-2}.$$

*Proof.* Fix $x \in X_p$ and put $t = d(x, p)$. The fact that $L(x, \text{id}) \le 1/t^2$ follows easily from the inequality $d_p(x, y) \le i_p(x, y) = d(x, y)/[d(x, p)d(y, p)]$. The lower bound $l(x, \text{id}) \ge 1/4t^2$ follows similarly from the inequality $d_p(x, y) \ge i_p(x, y)/4$, but we need to estimate a little more carefully to get rid of the 4.

Let $0 < \varepsilon < \frac{1}{8}$ and assume $0 < d(x, y) < \varepsilon^2 t$. Notice that

$$\frac{d(x, y)}{(1 + \varepsilon)^2 t^2} < \frac{d(x, y)}{(1 + \varepsilon) t^2} < \frac{\varepsilon^2 t}{(1 + \varepsilon) t^2} < \frac{\varepsilon}{8t} < \frac{1}{8t}.$$

Let $x = x_0, \dots, x_k = y \in X_p$. If $d(x_i, p) \le (1 + \varepsilon)t$ for all $0 \le i \le k$, then

$$\sum_{i=1}^{k} i_p(x_i, x_{i-1}) \ge \sum_{i=1}^{k} \frac{d(x_i, x_{i-1})}{(1 + \varepsilon)^2 t^2} \ge \frac{d(x, y)}{(1 + \varepsilon)^2 t^2}.$$

Suppose there exists $j \in \{1, \dots, k - 1\}$, such that $d(x_j, p) > (1 + \varepsilon)t$. Then

$$\sum_{i=1}^{k} i_p(x_i, x_{i-1}) \ge \sum_{i=1}^{j} i_p(x_i, x_{i-1}) \ge d_p(x, x_j) \ge 1/4 i_p(x, x_j).$$

If $d(x_j, p) > 2t$, then $d(x, x_j) > t$, so $d(x_j, p) \le 2d(x, x_j)$; therefore $i_p(x, x_j)/4 \ge 1/8t$. On the other hand, if $d(x_j, p) \le 2t$, then, since $d(x, x_j) \ge \varepsilon t$,

$$\frac{1}{4} i_p(x, x_j) = \frac{1}{4} \frac{d(x, x_j)}{t d(x_j, p)} \ge \frac{1}{4} \frac{\varepsilon t}{t d(x_j, p)} \ge \frac{\varepsilon}{8t}.$$

Thus in all cases

$$\sum_{i=1}^{k} i_p(x_i, x_{i-1}) \ge \frac{d(x, y)}{(1 + \varepsilon)^2 t^2};$$

so

$$\frac{d_p(x, y)}{d(x, y)} \ge \frac{1}{(1 + \varepsilon)^2 t^2}.$$

Letting $\varepsilon \to 0$, we obtain $l(x, \text{id}) \ge 1/t^2$, as required. □

The result above (in conjunction with Fact 4.3(a) and [14, 5.3]) tells us how to calculate the length of a path in $\mathrm{Inv}_p(X)$; that is, for each rectifiable path $\gamma$ in $X_p$ we have

$$\ell_p(\gamma) = \int_\gamma \frac{|dx|}{d(x,p)^2}. \tag{4.1}$$

In other words, $|dx|_p = |dx|/d(x,p)^2$ where $|dx|$ and $|dx|_p$ denote the arclength differentials in $X$ and $\mathrm{Inv}_p(X)$ respectively. Below we provide a simple consequence of this fact. See Proposition 6.3 for more information regarding quasiconvexity and inversions.

***Proposition 4.2.*** *If $(X,d)$ is locally $c$-quasiconvex and $p \in X$, then $(X_p, d_p)$ and $(X, \hat{d}_p)$ are both locally $5c$-quasiconvex.*

*Proof.* Let $\varepsilon > 0$ be a small number to be chosen below. Fix $x \in X_p$. Assume $r = \varepsilon d(x,p)$ has the property that points in $B(x;r)$ are joinable by $c$-quasiconvex paths. Let $y$, $z \in B(x;r)$ and select such a path $\alpha$ joining $y$, $z$. Note that

$$d_p(y,z) \geq \frac{i_p(y,z)}{4} \geq \frac{d(y,z)}{4[(1+\varepsilon)d(x,p)]^2}.$$

Now each point $u$ on $\alpha$ is at most $\frac{1}{2}\ell(\alpha)$ from one of $y$ or $z$, so $d(u,x) \leq (c+1)r$ and thus

$$[1-\varepsilon(c+1)]d(x,p) = d(x,p) - (c+1)r \leq d(u,p) \leq [1+\varepsilon(c+1)]d(x,p).$$

Therefore

$$\ell_p(\alpha) \leq \frac{c\,d(y,z)}{([1-\varepsilon(c+1)]d(x,p))^2} \leq 4c\frac{[(1+\varepsilon)d(x,p)]^2}{([1-\varepsilon(c+1)]d(x,p))^2} d_p(y,z).$$

The desired result follows by choosing $\varepsilon$ small enough so that both $r = \varepsilon d(x,p)$ has the assumed property and also $((1+\varepsilon)/[1-\varepsilon(c+1)])^2 \leq \frac{5}{4}$.

The proof for $(X, \hat{d}_p)$ is similar and left for the reader. □

Note that when $X$ is unbounded, neither $\mathrm{Inv}_p(X)$ nor $\mathrm{Sph}_p(X)$ need be locally quasiconvex at $p'$ or $\hat{p}$. For example, when we sphericalize the curve $X = \{(x,y) \in \mathbb{R}^2 : y = \sqrt{x}\sin x,\ 0 \leq x < \infty\}$ (with its induced Euclidean distance) with respect to the origin $p = (0,0)$, we find that $\hat{d}_p((n\pi,0),\hat{p})$ is comparable with $1/n$, but for the associated length metric $\hat{l}_p$, we have $\hat{l}_p((n\pi,0),\hat{p}) \simeq 1/n^{1/2}$; both comparisons are with constants independent of $n \in \mathbb{N}$.

Here is some useful information from [18, Lemmas 5.3, 5.5].

***Facts 4.3.***

(a) *If* $X \xrightarrow{f} Y$ *is a homeomorphism and* $x \in X$ *is non-isolated, then*

$$L(x,f) = 1/l(f(x),f^{-1}) \quad \text{and} \quad l(x,f) = 1/L(f(x),f^{-1}),$$

*where we use the convention that* $1/0 = \infty$, $1/\infty = 0$.

(b) *Suppose* $X \xrightarrow{f} Y \xrightarrow{g} Z$ *are continuous and* $x$, $f(x)$ *are non-isolated points of* $X$, $Y$ *respectively. Then*

$$L(x,g\circ f) \le L(x,f)L(f(x),g),$$
$$l(x,g\circ f) \ge l(x,f)l(f(x),g),$$

*provided that the products are not of the form* $0\cdot\infty$ *or* $\infty\cdot 0$.

(c) *Suppose* $X$ *is* $c$*-quasiconvex and* $X \xrightarrow{f} Y$ *satisfies* $L(x,f) \le M$ *for all* $x \in X$. *Then* $f$ *is* $cM$*-Lipschitz.*

The following stretching estimates are an easy consequence of Lemma 2.1 in conjunction with the basic lower bound for quasihyperbolic distance.

***Lemma 4.4.*** *Suppose* $(X,d)$ *is a locally* $c$*-quasiconvex incomplete locally compact space. Then the identity map* $\mathrm{id} : (X,d) \to (X,k)$ *satisfies*

$$\forall\, x \in X: \qquad \frac{1}{\mathrm{d}(x,\partial X)} \le l(x,\mathrm{id}) \le L(x,\mathrm{id}) \le \frac{c}{\mathrm{d}(x,\partial X)}.$$

***Corollary 4.5.*** *Suppose* $(X,d) \xrightarrow{f} (Y,d')$ *is a* $K$*-bilipschitz homeomorphism between locally* $c$*-quasiconvex incomplete locally compact spaces. Then as a map* $(X,k) \to (Y,k')$, $f$ *is* $cK^2$*-bilipschitz.*

*Proof.* Since $f$ and $f^{-1}$ are both $K$-Lipschitz (hence uniformly continuous), they have extensions to $\bar{X}$ and $\bar{Y}$ and we find that $f : (\bar{X},d) \to (\bar{Y},d')$ is $K$-bilipschitz. Moreover, for all $x \in X$, $K^{-1}\,\mathrm{d}(x,\partial X) \le \mathrm{d}'(fx,\partial Y) \le K\,\mathrm{d}(x,\partial X)$.

Let $(X,d) \xrightarrow{i} (X,k)$ and $(Y,d') \xrightarrow{j} (Y,k')$ be the indicated identity maps. Put $g = j\circ f\circ i^{-1}$. Since $(X,k)$ and $(Y,k')$ are geodesic, an appeal to Fact 4.3(c) reveals that we need only check that $L(x,g) \le cK^2$ and $L(y,g^{-1}) \le cK^2$ for $x \in X$ and $y \in \bar{Y}$ respectively. Using Facts 4.3(a,b) and Lemma 4.4 we find that

$$\begin{aligned} L(x,g) &\le L(f(x),j)\cdot L(x,f)\cdot L(x,i^{-1}) \\ &\le \frac{c}{\mathrm{d}'(f(x),\partial Y)}\cdot K\cdot \mathrm{d}(x,\partial X) \le cK^2. \end{aligned}$$

Similarly, $L(y,g^{-1}) \le cK^2$. ◻

***4.B. Inversions are quasihyperbolically bilipschitz.*** Let $(X, d)$ be complete and fix $p \in X$. Suppose $\Omega \subset X_p$ is a locally compact, open, rectifiably connected subspace of $X$. Then $(\mathrm{Inv}_p(X), d_p)$ is also complete and $I_p(\Omega)$ is a locally compact, open, rectifiably connected subspace of $\mathrm{Inv}_p(X)$. We further assume that when $\Omega$ is bounded, $\partial\Omega \neq \{p\}$, so $\partial_p\Omega \neq \emptyset$. We denote the quasihyperbolic metric in $I_p(\Omega)$ by $k_p$. (The reader should perhaps review Subsection 3.D.)

We demonstrate that inversions are bilipschitz with respect to quasihyperbolic distances, and we provide explicit quantitative bounds on the bilipschitz constant in various cases.

***Theorem 4.6.*** *Let $(X, d)$ be complete and fix a base point $p \in X$. Suppose $\Omega \subset X_p$ is a locally compact, open, locally $c$-quasiconvex subspace with $\partial\Omega \neq \emptyset \neq \partial_p\Omega$. Then the identity map $(\Omega, k) \xrightarrow{\mathrm{id}} (\Omega, k_p)$ is $M$-bilipschitz, where $M = 2c(a \vee 20b)$,*

$$a = \begin{cases} 1 & \text{if } \Omega \text{ is unbounded,} \\ \dfrac{\operatorname{diam}\Omega}{[\mathrm{d}(p, \partial\Omega) \vee (\operatorname{diam}\partial\Omega/2)]} & \text{if } \Omega \text{ is bounded.} \end{cases}$$

*and*

$$b = \begin{cases} b' & \text{if } X \text{ is } b'\text{-quasiconvex,} \\ 1 & \text{if } p \in \partial\Omega, \\ \dfrac{2\,\mathrm{d}(p, \partial\Omega)}{\mathrm{d}(p, \Omega)} & \text{if } p \notin \bar{\Omega}. \end{cases}$$

*Proof.* Here we refer to Facts 4.3(a, b, c) simply as (a), (b), (c). Since $(\Omega, k)$ and $(\Omega, k_p)$ are geodesic spaces, (c) tells us it suffices to check that

$$\forall\, x \in \Omega : \quad L(x, \mathrm{id}) \le M \quad \text{and} \quad L(x, \mathrm{id}^{-1}) \le M.$$

Let $(\Omega, d) \xrightarrow{h} (\Omega, d_p)$, $(\Omega, d) \xrightarrow{i} (\Omega, k)$, $(\Omega, d_p) \xrightarrow{j} (\Omega, k_p)$, denote the respective identity maps. Then $\mathrm{id} = j \circ h \circ i^{-1}$ and $\mathrm{id}^{-1} = i \circ h^{-1} \circ j^{-1}$. Thus (b) yields

$$L(x, \mathrm{id}) \le L(x, j) \cdot L(x, h) \cdot L(x, i^{-1}),$$
$$L(x, \mathrm{id}^{-1}) \le L(x, i) \cdot L(x, h^{-1}) \cdot L(x, j^{-1}).$$

According to Proposition 4.1, in conjunction with (a), we have

$$L(x, h) = \frac{1}{d(x, p)^2} \quad \text{and} \quad L(x, h^{-1}) = d(x, p)^2.$$

Since $\Omega$ is locally $c$-quasiconvex, Proposition 4.2 says $I_p(\Omega)$ is locally $5c$-quasiconvex. An appeal to Lemma 4.4 together with (a) now produces

$$L(x,i) \le \frac{c}{\delta(x)}, \qquad L(x,i^{-1}) \le \delta(x),$$

$$L(x,j) \le \frac{5c}{\delta_p(x)}, \quad L(x,j^{-1}) \le \delta_p(x).$$

The observations above reveal that it suffices to demonstrate that

$$\forall\, x \in \Omega: \quad \delta(x) \simeq \delta_p(x) d(x,p)^2;$$

more precisely, we must establish

$$\forall\, x \in \Omega: \quad 5c\delta(x) \le M\delta_p(x)d(x,p)^2 \quad \text{and} \quad c\delta_p(x)d(x,p)^2 \le M\delta(x).$$

Recalling the definition of $M$ and the estimates from Lemma 3.6, we see that the above inequalities are equivalent to

$$\forall\, x \in \Omega: \quad \delta_p(x) \le \frac{a}{d(x,p)} \quad \text{and} \quad \delta(x) \le bd(x,p).$$

Let $x \in \Omega$ be arbitrary.

We begin by showing that $\delta(x) \le bd(x,p)$. This is clear if $p \in \partial\Omega$, and also evident when $X$ is $b'$-quasiconvex (look at a $b'$-quasiconvex path joining $x$ to $p$). Suppose $p \notin \bar{\Omega}$. Select $q \in \partial\Omega$ with $d(p,q) \le \mathrm{d}(p,\partial\Omega) + \varepsilon$, where $\varepsilon > 0$ is small. Since $\mathrm{d}(p,\partial\Omega) \ge \mathrm{d}(p,\Omega)$, we may assume that $\delta(x) \ge 2d(x,p)$. Then

$$2d(x,p) \le \delta(x) \le d(x,q) \le d(x,p) + d(p,q),$$

so

$$d(p,q) \ge d(x,p)$$

and hence

$$\begin{aligned}\delta(x) \le d(x,q) \le 2d(p,q) &\le 2\frac{\mathrm{d}(p,\partial\Omega)+\varepsilon}{\mathrm{d}(p,\Omega)}\,\mathrm{d}(p,\Omega) \\ &\le 2\frac{\mathrm{d}(p,\partial\Omega)+\varepsilon}{\mathrm{d}(p,\Omega)} d(x,p).\end{aligned}$$

Letting $\varepsilon \to 0$ we obtain $\delta(x) \le bd(x,p)$ as asserted.

Our proof for the case when $\Omega$ is unbounded is now complete, because in this situation we have $p' \in \partial_p\Omega$ and then (3.3) provides the desired estimate

$\delta_p(x) \le 1/d(x,p)$. (☺) Thus we assume that $\Omega$ is bounded. (Recall that in this case we have the additional assumption $\partial\Omega \ne \{p\}$ to ensure that $\partial_p\Omega \ne \emptyset$.) We seek the estimate $\delta_p(x) \le a/d(x,p)$.

Select $q \in \partial\Omega$ so that $d(p,q) \ge \lambda \operatorname{diam} \partial\Omega$, where $0 < \lambda < \frac{1}{2}$. Then

$$\delta_p(x) \le d_p(x,q) \le \frac{d(x,q)}{d(q,p)} \frac{1}{d(x,p)} \le \frac{\operatorname{diam} \Omega}{d(q,p)} \frac{1}{d(x,p)}.$$

We now have three possibilities. If $\operatorname{diam} \partial\Omega = 0$, then $\partial\Omega = \{q\}$ and the above inequality becomes

$$\delta_p(x) \le \frac{\operatorname{diam} \Omega}{\mathrm{d}(p,\partial\Omega)} \frac{1}{d(x,p)}.$$

On the other hand, if $\operatorname{diam} \partial\Omega > 0$, then—by our choice of $q$ and letting $\lambda \to \frac{1}{2}$ —our initial inequality yields

$$\delta_p(x) \le 2\frac{\operatorname{diam} \Omega}{\operatorname{diam} \partial\Omega} \frac{1}{d(x,p)}.$$

Finally, if $\operatorname{diam} \partial\Omega > 0$ but $p \notin \bar{\Omega}$, then our previous inequality is also in force because $d(p,q) \ge \mathrm{d}(p,\partial\Omega)$. □

We remark that in general the bilipschitz constant $M$ in Theorem 4.6 may depend on the indicated data. Before exhibiting our examples, we point out that the ideas from our proof yield the following: If the identity map $(\Omega,k) \to (\Omega,k_p)$ is $M$-bilipschitz, then

$$\forall\, x \in \Omega: \quad \frac{1}{cM}\delta(x) \le \delta_p(x) d(x,p)^2 \le 5cM\delta(x).$$

In the examples below we use complex variables notation, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is the open unit disk in the complex number field $\mathbb{C}$ and $\mathbb{T} = \partial\mathbb{D}$ is the unit circle. We also take advantage of the fact, mentioned in the introduction, that Euclidean inversion enjoys the property $d_p = i_p$. First we show that $M$ may depend on the ratio $\operatorname{diam} \Omega / \operatorname{diam} \partial\Omega$.

***Example 4.7.*** Let $X = \bar{\mathbb{D}}$ be the closed unit disk in $\mathbb{C}$ (with Euclidean distance) and consider $\Omega = X \setminus \{p,q\}$ where $p = 0$ is the origin and $q = -t$ with $t \in (0,1)$. Then $\operatorname{diam} \Omega = 2$, $\operatorname{diam} \partial\Omega = t$, and for points $x \in [1-t, 1) \subset \Omega$ we find that

$$\delta(x) = d(x,p) = x\,,$$

$$\delta_p(x) = d_p(x,q) = i_p(x,q) = \frac{x+t}{xt} \ge \frac{1}{xt}.$$

Therefore, if the identity map $(\Omega, k) \to (\Omega, k_p)$ is $M$-bilipschitz, then

$$5Mx = 5M\delta(x) \geq \delta_p(x)d(x,p)^2 \geq \frac{x}{t},$$

and so

$$M \geq \frac{1}{5t} = \frac{1}{10}\frac{\operatorname{diam}\Omega}{\operatorname{diam}\partial\Omega}.$$

Next we show that $M$ may depend on the ratio $\operatorname{diam}\Omega / \operatorname{d}(p, \partial\Omega)$.

***Example 4.8.*** Let $X = \mathbb{T}$ be the unit circle, $t \in (0, \pi/2)$ and

$$\Omega = \{e^{i\theta} : t < \theta < 2\pi - t\}.$$

Then $\Omega$ is locally $(\pi/2)$-quasiconvex. Put $p = 1 \in X$, $q = e^{it}$ and $x = -1$. Then $\operatorname{diam}\Omega = 2$, $\operatorname{d}(p, \partial\Omega) = |p - q|$, $\delta(x) = |q + 1|$, $d(x,p)^2 = 4$, and

$$\delta_p(x) = d_p(x,q) = i_p(x,q) = \frac{|q+1|}{2|p-q|}.$$

Therefore if the identity map $(\Omega, k) \to (\Omega, k_p)$ is $M$-bilipschitz, then

$$\frac{5\pi}{2}M|q+1| = 5cM\delta(x) \geq \delta_p(x)d(x,p)^2 = \frac{4|q+1|}{2\,|p-q|},$$

and so

$$M \geq \frac{2}{5\pi}\frac{\operatorname{diam}\Omega}{\operatorname{d}(p,\partial\Omega)}.$$

Last we show that $M$ may depend on the ratio $\operatorname{d}(p, \partial\Omega) / \operatorname{d}(p, \Omega)$.

***Example 4.9.*** Fix $0 < t < 1$, put $X = [-1, 1] \cup [ti, 2i] \subset \mathbb{C}$, let $p = 2ti \in X$, and consider $\Omega = (-1, 1) \subset X$. Then $d(p, \partial\Omega) = \sqrt{1 + 4t^2}$ and $\operatorname{d}(p, \Omega) = 2t$. Taking $x = 0$, we find that

$$\delta(x) = 1, \quad d(x,p) = 2t, \quad \delta_p(x) = d_p(x,1) = i_p(x,1) = (2t\sqrt{1+4t^2})^{-1}$$

and thus if the identity map $(\Omega, k) \to (\Omega, k_p)$ is $M$-bilipschitz, then

$$1 = \delta(x) \leq M\delta_p(x)d(x,p)^2 = M\frac{4t^2}{2t\sqrt{1+4t^2}},$$

and so

$$M \geq \frac{\sqrt{1+4t^2}}{2t} = \frac{\operatorname{d}(p,\partial\Omega)}{\operatorname{d}(p,\Omega)}.$$

Now we take an attentive look at Theorem 4.6 in the special case where $\Omega$ is bounded and $p \in \partial\Omega$; here our identity map is $M$-bilipschitz with $M = 4c[10 \vee (\operatorname{diam}\Omega/\operatorname{diam}\partial\Omega)]$. The point of the following result is that the constant $L$ depends only on $c$, while the just mentioned bilipschitz constant $M$ in general depends on both $c$ and $\operatorname{diam}\Omega/\operatorname{diam}\partial\Omega$.

***Corollary 4.10.*** *Let $(X,d)$ be proper and fix a base point $p \in X$. Suppose $\Omega \subset X_p$ is an open locally $c$-quasiconvex subspace with* $\operatorname{diam}\partial\Omega > 0$ *(so, $\partial\Omega \neq \emptyset \neq \partial_p\Omega$). Assume further that $\Omega$ is bounded and $p \in \partial\Omega$. Then for all $x, y \in \Omega$,*

$$k_p(x,y) \le 40ck(x,y) + D_p\,,$$
$$k(x,y) \le 10ck_p(x,y) + D$$

*where $D = \operatorname{diam}_k(A)$, $D_p = \operatorname{diam}_{k_p}(A)$ and $A = \{z \in \Omega : d(z,p) \ge 2\operatorname{diam}\partial\Omega\}$. In particular, the map* $\operatorname{id} : (\Omega,k) \to (\Omega,k_p)$ *is an $(L,C)$-quasiisometry, where $L = 40c$ and $C = D_p \vee (D/10c)$.*

*Proof.* As noted above, we may assume that $\operatorname{diam}\Omega \ge 10\operatorname{diam}\partial\Omega$ (for otherwise id is $40c$-bilipschitz). Set $B = B(p; 2\operatorname{diam}\partial\Omega)$; then $\partial\Omega \subset B$ and $A = \Omega \setminus B$ is a non-empty compact subset of $\Omega$ (so $D$ and $D_p$ are finite). For $x \in B \cap \Omega$ we have $\delta_p(x) \le 5/d(x,p)$. This, together with a careful reading of the proof of Theorem 4.6, reveals that

$$\forall\, x \in B \cap \Omega: \quad L(x,\operatorname{id}) \le 40c \quad \text{and} \quad L(x,\operatorname{id}^{-1}) \le 10c,$$

which in turns implies that for each rectifiable arc $\alpha$ in $B \cap \Omega$,

$$(10c)^{-1}\ell_k(\alpha) \le \ell_{k_p}(\alpha) \le 40c\,\ell_k(\alpha).$$

Fix two points $x, y \in \Omega$. Assume at first that $x, y \in B$. Let $\gamma$ be an oriented $k$-geodesic from $x$ to $y$. Denote by $x'$ and $y'$ the first and last points of $\gamma$ in $A$. Since the subarcs $\gamma[x,x']$ and $\gamma[y,y']$ lie in $B \cap \Omega$, it follows that

$$k_p(x,x') \le \ell_{k_p}(\gamma[x,x']) \le 40c\,\ell_k(\gamma[x,x']) = 40ck(x,x');$$

similarly, $k_p(y,y') \le 40ck(y,y')$ (and clearly $k_p(x',y') \le D_p$). Consequently,

$$\begin{aligned} k_p(x,y) &\le k_p(x,x') + k_p(x',y') + k_p(y',y) \\ &\le 40c[k(x,x') + k(y',y)] + D_p \le 40ck(x,y) + D_p. \end{aligned}$$

Notice that if $x$ (or $y$) lies in $A$, then the inequalities above continue to hold provided we put $x' = x$ (or $y' = y$, or both).

Employing the same argument, but starting with a $k_p$-geodesic from $x$ to $y$, we obtain the inequality $k(x,y) \le 10ck_p(x,y) + D$. □

In general, quantitative bounds for the constants $D$ and $D_p$ above are not available; indeed, simple examples reveal that it is possible for $D$ (and so $C$ too) to be arbitrarily large if $\operatorname{diam}\Omega/\operatorname{diam}\partial\Omega$ is large. However, from the above proof we obtain the following estimates which are valid for all $x, y \in \Omega \cap B(p; 2\operatorname{diam}\partial\Omega)$:

$$k_p(x,y) \le 40ck(x,y) + \operatorname{diam}_{k_p}(S),$$

$$k(x,y) \le 10ck_p(x,y) + \operatorname{diam}_k(S),$$

where $S = \{z \in \Omega : d(z,p) = 2\operatorname{diam}\partial\Omega\}$. The point here is that when $X$ is annular quasiconvex (see Section 6) we can find estimates for $\operatorname{diam}_k(S)$ and $\operatorname{diam}_{k_p}(S)$ in terms of the associated data.

We close this subsection by reporting that the sphericalization construction of Bonk and Kleiner also produces a quasihyperbolically bilipschitz identity map. We write $\hat{k}_p$ for the quasihyperbolic metric in $S_p(\Omega)$. (The reader should perhaps review Subsection 3.D.)

***Theorem 4.11.*** *Let $(X,d)$ be complete and fix a base point $p \in X$. Suppose $\Omega \subset X$ is a locally compact, open, locally $c$-quasiconvex subspace with $\partial\Omega \ne \emptyset$. Then the identity map* $\mathrm{id} : (\Omega,k) \to (\Omega,\hat{k}_p)$ *is bilipschitz. When $\Omega$ is unbounded and $p \in \partial\Omega$, we get the bilipschitz constant* $80c$.

*Proof.* The general result follows from Theorem 4.6 since sphericalization is a special case of inversion. Indeed, as explained at the end of Subsection 3.B, we have $(\operatorname{Sph}_p(X), \hat{d}_p)$ isometric to $(\operatorname{Inv}_q(X^q), (d^{p,q})_q)$. Note that distance to the boundary of $\Omega$ is the same regardless of which ambient space ($X$ or $X^q$) we consider, so there is only one quasihyperbolic distance associated with $(\Omega,d) = (\Omega,d^{p,q})$. Also, the quasihyperbolic distance in $I_q(\Omega)$ is just $\hat{k}_p$. Finally, when $\Omega$ is unbounded and $p \in \partial\Omega$, our bilipschitz constant $M = 2c(a \vee 20b)$ is $80c$ since $a = 1$ and $b = 2$. (In fact, the careful reader will notice that, since $\delta(x) \le d(x,p) < d^{p,q}(x,q)$, the proof of Theorem 4.6 actually provides the bilipschitz constant $40c$.) ◻

## 5. INVERSIONS AND UNIFORMITY

Here we demonstrate that both inversion and sphericalization preserve the class of uniform subspaces; see Theorems 5.1, 5.5. We also provide quantitative information describing precisely how the new uniformity constants depend on the associated data.

***5.A. Uniform subspaces.*** Roughly speaking, a space is uniform provided points in it can be joined by paths which are not too long and which stay away from the region's boundary (so-called quasiconvex twisted double cone arcs). Uniform domains in Euclidean space were first studied by John and Martio and Sarvas who proved injectivity and approximation results for them. They are well recognized as being the nice domains for quasiconformal function theory as well as many other areas of geometric analysis (e.g., potential theory); see [6] and [16] for various references. Every (bounded) Lipschitz domain is uniform, but generic uniform domains may very well have fractal boundary. Recently, uniform subdomains of Heisenberg groups, as well as more general Carnot groups, have become a focus of study. Bonk, Heinonen and Koskela [3] introduced the notion of uniformity in the general metric space setting.

Before we continue, we comment on the difference between our definition of uniformity and the one seen elsewhere in the metric space literature, notably in [3]. The usual definition involves a given locally compact rectifiably connected incomplete space $(U, d)$, whose boundary is defined to consist of all points in $\bar{U} \setminus U$. We instead consider locally compact rectifiably connected open sets $\Omega$ in a complete ambient space $X$ and define the boundary of $\Omega$ to be its topological boundary (which is assumed to be nonempty). In particular, if $\Omega$ is dense in $X$, then our definition reduces to the usual one, with the pair $(\Omega, X)$ playing the role of $(U, \bar{U})$. The added generality of allowing $X \setminus \bar{\Omega}$ to be nonempty is irrelevant if we only want to consider uniformity, but it is useful in our discussion as it allows us to examine a wider class of inversions with respect to base points $p \in X \setminus \Omega$. Note that our invariance results are a little different depending on whether $p \in \partial\Omega$ or $p \in X \setminus \bar{\Omega}$; see Theorems 5.1, 5.5 and Examples 5.2, 5.3, 5.4.

Let $(X, d)$ be a complete metric space. Suppose $\Omega \subset X$ is a locally compact open subspace with $\partial\Omega \neq \emptyset$. Recall that $\delta(x)$ denotes the distance (in $(X, d)$) from a point $x$ to the boundary of $\Omega$. We call $\gamma : [0, 1] \to \Omega$ a *c-uniform path*, $c \geq 1$, provided

$$\ell(\gamma) \leq c d(\gamma(0), \gamma(1)),$$

$$\forall\, t \in [0, 1] : \quad \ell(\gamma|_{[0,t]}) \wedge \ell(\gamma|_{[t,1]}) \leq c\delta(\gamma(t)).$$

If every pair of points in $\Omega$ can be joined by a $c$-uniform path, we dub $\Omega$ a *c-uniform* subspace of $X$. The first condition on $\gamma$ is called the *quasiconvexity* condition, and the second is the *double cone arc* condition. Note that the existence of uniform paths implies the existence of uniform arcs: we simply cut out any loops, a process that preserves both the quasiconvexity and double cone conditions; see [17].

An especially important property of uniform spaces is that quasihyperbolic geodesics in a $c$-uniform space are $b$-uniform arcs where $b = b(c)$ depends only on $c$ (e.g., we can take $b = \exp(1000c^6)$). (See [7, Theorems 1,2] for domains in Euclidean space and [3, Theorem 2.10] for general metric spaces.) Because of this, we may—and from now on will—assume that all quasihyperbolic geodesics in a $c$-uniform space are $c$-uniform arcs. Quasihyperbolic geodesics in uniform spaces constitute a special class of uniform arcs: any subarc of such an arc is also a uniform arc with the same uniformity constant. This property is not shared by general uniform paths.

We need to know that boundary points in a locally compact uniform space can be joined by quasihyperbolic geodesics, and that these geodesics are still uniform arcs. A routine application of the Arzela-Ascoli theorem establishes the existence of quasihyperbolic geodesic rays joining an interior point to a boundary point (however, the construction for joining two boundary points with a quasihyperbolic geodesic line is a tad more delicate) and this is all we require. It is easy to verify that these limit geodesics are uniform arcs.

***5.B. Main results and examples.*** Here we present Theorems 5.1 and 5.5 which say that inversions and sphericalizations preserve the class of uniform subspaces of a complete ambient space. We provide explicit estimates for the new uniformity constants which depend on the original uniformity constant and also on other quantities in various cases. As in Subsection 3.D, $\Omega$ and $I_p(\Omega)$ are equal as sets, but the former has the metric $d$ attached, while the latter has the metric $d_p$. Recall that $\partial\Omega_p = \partial\Omega \setminus \{p\}$ when $\Omega$ is bounded, but $\partial\Omega_p = (\partial\Omega \setminus \{p\}) \cup \{p'\}$ if $\Omega$ is unbounded.

Below we employ the following notation for $\Omega \subset X_p$ with $\partial_p\Omega \neq \emptyset$:

$$b(p) := \sup\{d(p,q) : q \in \partial\Omega\}$$

and

$$r(p) := \begin{cases} \mathrm{d}(p,\partial\Omega)/\,\mathrm{d}(p,\Omega) & \text{if } p \in X \setminus \bar{\Omega}, \\ 0 & \text{if } p \in \partial\Omega. \end{cases}$$

We only require $b(p)$ when $\Omega$ is bounded. If $\partial\Omega$ has at least two points, $b(p) \geq \operatorname{diam}(\partial\Omega)/2$; if $\partial\Omega = \{q\}$, $b(p) = d(p,q) > 0$. Also: when $p \in X \setminus \bar{\Omega}$, $r(p) \geq 1$; if $X$ is $a$-quasiconvex, then $r(p) \leq a$ and so in this case $c' = c'(c,a)$ below in 5.1(a).

***Theorem 5.1.*** *Let $(X,d)$ be a complete metric space and fix $p \in X$. Suppose $\Omega \subset X_p$ is open and locally compact with $\partial\Omega \neq \emptyset \neq \partial_p\Omega$. Then $\Omega$ is uniform if and only if $I_p(\Omega)$ is uniform. More precisely:*

(a) *If $\Omega$ is $c$-uniform, then $I_p(\Omega)$ is $c'$-uniform, where $c' = c_0[1 + r(p)]^4$ and $c_0$ depends only on $c$. For instance, we may take $c_0 = 3^5 2^{13} c^6 (c+1)^2$.*

(b) *If $I_p(\Omega)$ is $c$-uniform and $\Omega$ unbounded, then $\Omega$ is $c''$-uniform, where $c'' = 256c_0$ and $c_0$ is as in* (a).

(c) *If $I_p(\Omega)$ is $c$-uniform and $\Omega$ bounded, then $\Omega$ is $c''$-uniform, where $c'' = c_0 \operatorname{diam}(\Omega)/b(p)$, and $c_0$ depends only on $c$. For instance, we may take $c_0 = 3 \cdot 2^{16} c^2 (8c+1)^2$.*

See 5.7, 5.8, 5.10 for the proofs of Theorem 5.1(a,b,c) respectively. We will see in our proof of 5.1(c) that only the quasiconvexity parameter may depend on $\operatorname{diam}(\Omega)/b(p)$; the double cone arc parameter depends only on $c$.

Before continuing, we consider some examples which show that the uniformity constants $c'$ in the above theorem may depend not only on $c$, but also on the other indicated parameters. As noted in the introduction (for $p = 0$), inversion in $\mathbb{R}^n$ with respect to the base point $p$ gives the pullback metric associated with the self-homeomorphism $x \mapsto x/|x-p|^2$ of $\mathbb{R}^n \setminus \{p\}$. We use this characterization for the three examples below in dimension $n = 2$. As in Examples 4.7, 4.8, 4.9, we identify $\mathbb{R}^2$ with the complex number field $\mathbb{C}$ and use complex variables notation.

***Example 5.2.*** Fix $0 < t < 1$, put

$$X = [-1,1] \cup [ti, 2i] = \{x : x \in \mathbb{R},\ |x| \le 1\} \cup \{iy : y \in \mathbb{R},\ t \le y \le 2\} \subset \mathbb{C},$$

set $p = 2ti \in X$, and let $\Omega = (-1,1) \subset X$. Clearly $\Omega$ is 1-uniform. Consider the points $u = -1 + t$, $v = 1 - t \in \Omega$. As $t \to 0$, we see that $i_p(u,v) \to 2$, so $\limsup_{t\to 0} d_p(u,v) \le 2$. However, any path from $u$ to $v$ in $\Omega$ must pass through the point $0 \in \Omega$, and so when $t \to 0$ its $d_p$-length will get arbitrarily large. Thus $c'$ may depend on $\mathrm{d}(p,\partial\Omega)/\mathrm{d}(p,\Omega)$ in 5.1(a). Notice that as $t \to 0$, $\mathrm{d}(p,\partial\Omega) = \sqrt{1+4t^2} \to 1$ whereas $\mathrm{d}(p,\Omega) = 2t \to 0$.

***Example 5.3.*** Let $X = \mathbb{T} = \partial\mathbb{D}$ be the unit circle in $\mathbb{C}$ with Euclidean distance. Fix $t \in (0, \pi/2)$ and consider $\Omega = \{e^{i\theta} : t < \theta < 2\pi - t\}$. The uniformity constant of $\Omega$ is very large for very small $t > 0$, since $|z - w|$ is very small for the points $z = e^{2it}$ and $w = e^{-2it}$, but a path from $z$ to $w$ in $\Omega$ has length at least $2(\pi - 2t)$, i.e., close to $2\pi$. Since $\Omega$ is an open segment on a circle through $p = 1$, $I_p(\Omega)$ is isometric to an open interval, and so is a 1-uniform space irrespective of the value of $t \in (0,1)$. Thus $c''$ may depend on $\mathrm{diam}(\Omega)/b(p)$ in 5.1(c); here $\mathrm{diam}(\Omega) = 2$ but $b(p) = 2\sin(t/2) \to 0$ as $t \to 0^+$.

***Example 5.4.*** Fix $0 < t < \pi/2$, set $p = e^{it}$, $q = e^{2it}$ and let $\Omega = \{e^{i\theta} : 2t < \theta \le 2\pi\}$ viewed as a domain in the proper metric space

$$X = \{e^{i\theta} : 2t \le \theta \le 2\pi\} \cup \{p\} \subset \mathbb{T} \subset \mathbb{C}.$$

The sole boundary point of $\Omega$ is $q$. As before, $I_p(\Omega)$ is isometric to an interval, and so is a 1-uniform space for any $t \in (0,1)$. Also, any uniformity constant $c''$ for $\Omega$ must be very large for very small $t > 0$. Thus $c''$ may depend on $\mathrm{diam}(\Omega)/d(p,q)$ in 5.1(c) when $\Omega$ is bounded and $\partial\Omega = \{q\}$. Again $\mathrm{diam}(\Omega) = 2$, but $d(p,q) = 2\sin(t/2) \simeq t$ as $t \to 0^+$.

We next derive a consequence of Theorem 5.1. As in Subsection 3.D, $\Omega$ and $S_p(\Omega)$ are equal as sets, but the former has the metric $d$ attached, while the latter has the metric $\hat{d}_p$.

***Theorem 5.5.*** *Let $(X,d)$ be an unbounded complete metric space, let $\Omega \subset X$ be open and locally compact with $\partial\Omega \ne \emptyset$, and let $p \in X$. Then $\Omega$ is uniform if and only if $S_p(\Omega)$ is uniform. Moreover:*

(a) *If* $\Omega$ *is* $c$*-uniform, then* $S_p(\Omega)$ *is* $\hat{c}$*-uniform, where* $\hat{c} = \hat{c}_0 \cdot (1 + 2\,\mathrm{d}(p, \partial\Omega))$ *and* $\hat{c}_0 = \hat{c}_0(c)$.

(b) *If* $\Omega$ *is unbounded and* $S_p(\Omega)$ *is* $\hat{c}$*-uniform, then* $\Omega$ *is* $c$*-uniform, where* $c = c(\hat{c})$.

*Proof.* The general result follows from Theorem 5.1 since sphericalization is a special case of inversion (as explained at the end of Section 3.B). Let $Y = \mathrm{Sph}_p(X)$, let $d' = (\hat{d}_p)_{\hat{p}}$ be the metric on $\mathrm{Inv}_{\hat{p}}(Y)$, and recall from Proposition 3.4 that the identity map $(X, d) \to (\mathrm{Inv}_{\hat{p}}(Y), d') = (X, d')$ is 16-bilipschitz.

(a) Since $(\Omega, d)$ is $c$-uniform, $(\Omega, d')$ is $256c$-uniform. Select $q \in \partial\Omega$ with $d(p, q) \le 2\,\mathrm{d}(p, \partial\Omega)$. Then $\mathrm{diam}(Y) \le 1$, $\hat{d}_p(q, \hat{p}) \ge 1/[4(1 + 2\,\mathrm{d}(p, \partial\Omega))]$, and $q \in \hat{\partial}_p\Omega$. Theorem 5.1(c) applied to $(Y, \hat{d}_p)$ now implies that $S_p(\Omega)$ is $\hat{c}$-uniform, where $\hat{c} = 4c_0(1 + 2\,\mathrm{d}(p, \partial\Omega))$ and $c_0$ is as in Theorem 5.1(c) but with $c$ replaced by $256c$.

(b) Since $\hat{p} \in \hat{\partial}_p\Omega$, Theorem 5.1(a) applied to $(Y, \hat{d}_p)$ implies that $(\Omega, d')$ is $c'$-uniform with $c' = c'(\hat{c})$. Thus $(\Omega, d)$ is $256c'$-uniform. ◻

5.C. ***Proofs of Theorem* 5.1 (a) *and* (b).** We begin with a number of lemmas which are required for these promised proofs; the latter can be found in 5.7 and 5.8. In the remainder of this section we encounter various explicit constants $b_i$ and $c_i$, all of which are at least 1. The constants $c_i$ appear in statements of results and are never re-used whereas the $b_i$ appear in proofs and may be defined differently in different proofs.

First we exhibit some elementary inequalities. Suppose $\gamma$ is a rectifiable path in $X_p$ joining $x$ to $y$. Let

$$t = d(x, p), \quad s = d(y, p), \quad \text{and} \quad R = \frac{1}{2}[\ell(\gamma) + t + s].$$

Using the inequality

$$\forall\, u \in [0, \ell(\gamma)] : \quad [\ell(\gamma) - u + t] \wedge [u + s] \le R,$$

we easily deduce that

$$\forall\, z \in \gamma : \quad d(z, p) \le [\ell(\gamma[x, z]) + t] \wedge [\ell(\gamma[y, z]) + s] \le R; \tag{5.1}$$

when $\gamma$ is $c$-quasiconvex, we can take $R = \frac{1}{2}(c + 1)(t + s)$.

Next, a glance back at (4.1) reveals that

$$\forall \text{ rectifiable } \gamma \subset A(p; r, R) : \quad \frac{\ell(\gamma)}{R^2} \le \ell_p(\gamma) \le \frac{\ell(\gamma)}{r^2}. \tag{5.2}$$

***Lemma 5.6.*** *Let $(X,d)$ be a metric space, $p \in X$, $\Omega \subset X_p$ be open with $\partial\Omega \neq \emptyset$, and let $0 < r < R < \infty$. Suppose $\gamma$ is a path in $A(p;r,R) \cap \Omega$ joining $x$, $y$. Put $t = d(x,p)$, $s = d(y,p)$.*

(a) *If $\gamma$ is $c$-quasiconvex in $X$, then it is $(4cst/r^2)$-quasiconvex in $\mathrm{Inv}_p(X)$.*
(b) *If $\gamma$ is $c$-quasiconvex in $\mathrm{Inv}_p(X)$, then it is $(cR^2/st)$-quasiconvex in $X$.*
(c) *If $\gamma$ is $c$-uniform in $\Omega$, then it is $c_1$-uniform in $I_p(\Omega)$, where $c_1 = 8c(R/r)^2$.*
(d) *If $\gamma$ is $c$-uniform in $I_p(\Omega)$, then it is $c_2$-uniform in $\Omega$, where $c_2 = 2cR^2/r^2$.*

*Proof.* To prove (a), we use (5.2), quasiconvexity, and (3.1) to get

$$\ell_p(\gamma) \le \frac{\ell(\gamma)}{r^2} \le \frac{cd(x,y)}{r^2} \le \frac{4cst}{r^2} d_p(x,y).$$

We omit the similar argument for (b) and next prove (c). Since $\ell_p(\gamma) \le 2cR/r^2$, Lemma 3.6(a) and (5.2) produce

$$\begin{aligned} 8R\delta_p(z) &\ge \frac{r^2}{cR}\left(\frac{\ell_p(\gamma)}{2} \wedge \ell_p(\gamma[x,z]) \wedge \ell_p(\gamma[y,z])\right) \\ &\ge \frac{r^2}{cR}[\ell_p(\gamma[x,z]) \wedge \ell_p(\gamma[y,z])] \end{aligned}$$

which holds for every point $z$ on $\gamma$. For (d) we first use (b) to see that

$$\ell(\gamma) \le \frac{cR^2}{st} d(x,y) \le cR^2 \frac{s+t}{st} \le \frac{2cR^2}{r},$$

and then Lemma 3.6(b) and (5.2) produce

$$\begin{aligned} \frac{2}{r}\delta(z) &\ge \frac{r}{cR^2}\left(\frac{\ell(\gamma)}{2} \wedge \ell(\gamma[x,z]) \wedge \ell(\gamma[y,z])\right) \\ &\ge \frac{r}{cR^2}[\ell(\gamma[x,z]) \wedge \ell(\gamma[y,z])] \end{aligned}$$

which holds for every point $z$ on $\gamma$. □

5.7. ***Proof of Theorem 5.1* (a)** Let $x$, $y \in \Omega$, write $t = d(x,p)$, $s = d(y,p)$ and assume $t \le s$. Let $\gamma$ be a quasihyperbolic geodesic in $\Omega$ from $x$ to $y$; so $\gamma$ is a $c$-uniform arc in $\Omega$. We demonstrate that $\gamma$ is also a uniform arc in $I_p(\Omega)$. We consider the cases $s \le 8t$ and $s > 8t$.

***Case 1.*** $s \le 8t$. We claim that

$$\forall\, z \in \gamma : \quad \frac{t}{b_1} \le d(z,p) \le \frac{9}{2}(c+1)t,$$

where $b_1 = 2c[1 + r(p)]$. Since $\gamma$ is $c$-quasiconvex, (5.1) provides the upper bound. Establishing the lower bound is the only part of the proof where $r(p)$ enters the picture.

If $\ell(\gamma[x,z]) \le t/2$, then $d(z,x) \le t/2$ and $d(z,p) \ge d(x,p) - d(x,z) \ge t/2$. Similarly, if $\ell(\gamma[z,y]) \le t/2$, then $d(z,p) \ge t/2$. Suppose $\ell(\gamma[x,z]) \wedge \ell(\gamma[z,y]) \ge t/2$. If $p \in \partial\Omega$, then

$$d(z,p) \ge \delta(z) \ge \frac{[\ell(\gamma[x,z]) \wedge \ell(\gamma[z,y])]}{c} \ge \frac{t}{2c},$$

while if $p \in X \setminus \bar{\Omega}$, we replace the first inequality above by the inequality

$$[1 + r(p)]d(z,p) \ge d(z,p) + r(p)\,\mathrm{d}(p,\Omega) = d(z,p) + \mathrm{d}(p,\partial\Omega) \ge \delta(z).$$

Now Lemma 5.6(c), with $r = t/b_1$ and $R = 9(c+1)t/2$, implies that $\gamma$ is $b_2$-uniform, where $b_2 = 8cR^2/r^2 = 2 \cdot 3^4 c b_1^2 (c+1)^2 = 2^3 3^4 c^3 (c+1)^2 [1 + r(p)]^2$.

***Case 2.*** $s > 8t$. Let $n \ge 3$ be the integer with $2^n t < s \le 2^{n+1} t$. For each $i$, $1 \le i \le n$, fix some $x_i \in \gamma$ with $d(x_i, p) = 2^i t$. Also let $x_0 = x$, $x_{n+1} = y$ and put $\gamma_i = \gamma[x_{i-1}, x_i]$. Note that

$$\forall\, 1 \le i \le n: \quad \begin{aligned} d_p(x_{i-1}, x_i) &\le \frac{1}{d(x_{i-1},p)} + \frac{1}{d(x_i,p)} = \frac{3}{2^i t}, \\ d_p(x_n, y) &\le \frac{4}{2^{n+1} t}. \end{aligned}$$

We can apply Case 1 to each of the subarcs $\gamma_i$. In particular, since each $\gamma_i$ is $c$-quasiconvex in $\Omega$, Lemma 5.6(a) implies that $\gamma_i$ is $(8cb_1^2)$-quasiconvex in $I_p(\Omega)$. Thus

$$\ell_p(\gamma) = \sum_{i=1}^{n+1} \ell_p(\gamma_i) \le \frac{24cb_1^2}{t}.$$

Since $s > 8t$, $d(x,y) \ge s - t > 7s/8$, whence $d_p(x,y) \ge d(x,y)/4st \ge 7/32t$ and we conclude that $\gamma$ is $b_3$-quasiconvex with $b_3 = 3 \cdot 2^8 c b_1^2/7$.

Note that we can argue as above to obtain

$$\forall\, 1 \le j \le n+1: \quad \ell_p(\gamma[x_j, y]) \le \frac{24cb_1^2}{2^j t}.$$

It remains to show that $\gamma$ satisfies a double cone arc condition. We can apply Case 1 to each of the subarcs $\gamma[x_{i-2}, x_{i+1}]$ to see that these are $b_2$-uniform in $I_p(\Omega)$. Let $z \in \gamma$. Then $z \in \gamma_i$ for some $1 \le i \le n+1$. First we consider the 'end' cases $i \le 2$ or $i \ge n$.

Suppose $z \in \gamma_1 \cup \gamma_2$. Then

$$\ell_p(\gamma[x_3, z]) \geq \ell_p(\gamma_3) \geq \frac{1}{4} i_p(x_2, x_3) \geq \frac{1}{32t}$$

so

$$\ell_p(\gamma[y, z]) \leq \ell_p(\gamma) \leq \frac{24cb_1^2}{t} \leq 3 \cdot 2^8 cb_1^2 \ell_p(\gamma[x_3, z])$$

and thus

$$\begin{aligned}\ell_p(\gamma[x, z]) \wedge \ell_p(\gamma[y, z]) &\leq 3 \cdot 2^8 cb_1^2 [\ell_p(\gamma[x, z]) \wedge \ell_p(\gamma[x_3, z])] \\ &\leq 3 \cdot 2^8 cb_1^2 b_2 \delta_p(z)\end{aligned}$$

where the last inequality holds because $\gamma[x_0, x_3]$ is $b_2$-uniform in $I_p(\Omega)$.

Suppose $z \in \gamma_n \cup \gamma_{n+1}$. Then

$$\ell_p(\gamma[x_{n-2}, z]) \geq \ell_p(\gamma_{n-1}) \geq \frac{1}{4} i_p(x_{n-2}, x_{n-1}) \geq \frac{1}{2^{n+1} t}$$

so

$$\ell_p(\gamma[y, z]) \leq \ell_p(\gamma[x_{n-1}, y]) \leq \frac{24cb_1^2}{2^{n-1} t} \leq 3 \cdot 2^5 cb_1^2 \ell_p(\gamma[x_{n-2}, z])$$

and thus

$$\begin{aligned}\ell_p(\gamma[x, z]) \wedge \ell_p(\gamma[y, z]) &\leq 3 \cdot 2^5 cb_1^2 [\ell_p(\gamma[x_{n-2}, z]) \wedge \ell_p(\gamma[y, z])] \\ &\leq 3 \cdot 2^5 cb_1^2 b_2 \delta_p(z)\end{aligned}$$

where the last inequality holds because $\gamma[x_{n-2}, x_{n+1}]$ is $b_2$-uniform in $I_p(\Omega)$. We conclude that a double $b_4$-cone inequality holds when $z \in \gamma_1 \cup \gamma_2 \cup \gamma_n \cup \gamma_{n+1}$, where $b_4 = 3 \cdot 2^8 cb_1^2 b_2$.

Now we deal with the cases $3 \leq i < n$. Put $u = x_{i-2}$, $v = x_{i+1}$, $\alpha = \gamma[u, z]$, $\beta = \gamma[z, v]$ and remember that by Case 1, $\gamma[u, v]$ is $b_2$-uniform in $I_p(\Omega)$. Suppose $\ell_p(\alpha) \leq \ell_p(\beta)$. Then

$$\begin{aligned}b_2 \delta_p(z) \geq \ell_p(\alpha) &\geq \ell_p(\gamma[x_{i-2}, x_{i-1}]) \geq d_p(x_{i-2}, x_{i-1}) \\ &\geq \frac{d(x_{i-2}, x_{i-1})}{4d(x_{i-2}, p) d(x_{i-1}, p)} \geq \frac{1}{2^{i+1} t},\end{aligned}$$

so $\ell_p(\gamma[z, y]) \leq b_5 \delta_p(z)$, with $b_5 = 3 \cdot 2^5 cb_1^2 b_2$, and a double $b_5$-cone arc inequality holds. Suppose instead that $\ell_p(\beta) \leq \ell_p(\alpha)$. Then

$$b_2 \delta_p(z) \geq \ell_p(\beta) \geq \ell_p(\gamma[x_i, x_{i+1}]) \geq d_p(x_i, x_{i+1}) \geq \frac{1}{2^{i+3} t},$$

so $\ell_p(\gamma[z,y]) \le 4b_5\delta_p(z)$ and we get a double $4b_5$-cone arc inequality in this case.

Since $b_4 = 8b_5 > b_2$, we have a double $b_4$-cone arc inequality in all cases. Since $b_4 > b_3$, we have proved that $\gamma$ is a $c'$-uniform curve in $I_p(\Omega)$, where $c' = b_4$. □

5.8. ***Proof of Theorem* 5.1 (b)** Note that $p' \in \partial_p\Omega$ as $\Omega$ is unbounded. Thus, abusing notation, we have $r(p') = 0$ and—since $I_p(\Omega)$ is $c$-uniform—an appeal to Theorem 5.1(a) permits us to assert that $(\Omega, (d_p)_{p'})$ is $c_0$-uniform. Invoking Proposition 3.3 we conclude that $(\Omega, d)$ is $256c_0$-uniform. □

5.D. ***Proof of Theorem* 5.1 (c).** This proof is more difficult than for parts (a) and (b). It follows from the next proposition, whose proof (see 5.14) is delayed until we have proven some further lemmas. Everywhere below (in Proposition 5.9 and Lemmas 5.11, 5.12) we assume the following: $(X,d)$ is a complete metric space; $p \in X$; $\Omega \subset X_p$ is open, locally compact, and bounded with $\partial\Omega \ne \emptyset \ne \partial_p\Omega$; $b(p)$ is as defined at the beginning of Subsection 5.B; $k_p$ is the quasihyperbolic metric in $I_p(\Omega)$; and, of course, $I_p(\Omega)$ is $c$-uniform.

***Proposition 5.9.*** *Let $\gamma$ be a $k_p$-geodesic joining $x$, $y \in \Omega$ with $t = d(x,p) \le d(y,p)$. Then $\gamma$ is a $b$-uniform arc in $\Omega$. If $d(x,y) \le t/8c$, we can take $b = c_3 = 8c(8c+1)^2$; if $d(x,y) \ge t/8c$, then*

$$\ell(\gamma) \le c_4\rho d(x,y),$$
$$\forall\, z \in \gamma:\ \ell(\gamma[x,z]) \wedge \ell(\gamma[z,y]) \le c_5\delta(z),$$

*where $c_4 = 3\cdot 2^{14}c^3$, $\rho = \operatorname{diam}(\Omega)/b(p)$, $c_5 = 3\cdot 2^{16}c^2(8c+1)^2$ and so $b = (c_4\rho) \vee c_5$ works.*

5.10. ***Proof of* 5.1 (c).** According to Proposition 5.9, the $k_p$-geodesics are $b$-uniform arcs in $\Omega$, where $b = (c_4\rho) \vee c_5 \le c''$. □

***Lemma 5.11.*** *Suppose $\gamma$ is a $k_p$-geodesic between $x$, $y \in \Omega$, with $d(x,p) \le d(y,p)$. Then:*

(a) *For all $z \in \gamma$, $d(z,p) \ge d(x,p)/(8c+1)$.*
(b) *If there exists a number $K \ge 1$ such that $d(z,p) \le Kd(x,p)$ for all $z \in \gamma$, then $\gamma$ is $K^2c$-quasiconvex and $c_6$-uniform in $\Omega$, where $c_6 = 2K^2c(8c+1)^2$.*

*Proof.* Let $t = d(x,p)$ and $s = d(y,p)$. We first prove (a). Using (3.1) twice, plus the fact that $\gamma$ is $c$-quasiconvex in $I_p(\Omega)$, we get

$$\frac{d(x,z)}{4td(z,p)} \le d_p(x,z) \le \ell_p(\gamma) \le cd_p(x,y) \le \frac{2c}{t},$$

and so $d(x,z) \le 8cd(z,p)$. Thus $t \le d(x,z) + d(z,p) \le (8c+1)d(z,p)$.

As for (b), the hypotheses and part (a) imply that $\gamma \subset A(p;r,R)$, where $r = t/(8c+1)$ and $R = Kt$. The desired quasiconvexity and uniformity of $\gamma$ are thus given by Lemma 5.6(b,d), since $cR^2/st \le K^2c$ and $2cR^2/r^2 = c_6$. □

***Lemma 5.12.*** *Suppose $\gamma$ is a $k_p$-geodesic between $x, y \in \Omega$, where $d(x,p) \le d(y,p)/8$ and $d(y,p) \ge d(z,p)$ for all $z \in \gamma$. Then $\gamma$ is $c_7$-quasiconvex and $c_8$-uniform in $\Omega$, where $c_7 = 3 \cdot 2^5 c$ and $c_8 = 5 \cdot 2^{12} c^2 (8c+1)^2$.*

*Proof.* Let $t = d(x,p)$ and choose the integer $n \ge 2$ with $2^n t < d(y,p) \le 2^{n+1} t$. For $1 \le i \le n$, let $x_i$ be the first point on $\gamma$ (oriented from $x$ to $y$) with $d(x_i,p) = 2^i t$. Also set $x_0 := x$, $x_{n+1} := y$. Set $\gamma_i := \gamma[x_{i-1}, x_i]$, $1 \le i \le n+1$. By the choice of $x_i$, we have $d(z,p) \le 2^i t$ for all $z \in \gamma_i$. Applying Lemma 5.11(b) to each $\gamma_i$ (with $(2^{i-1}t, 2)$ taking the place of $(t,K)$), we see that

$$\ell(\gamma_i) \le 8cd(x_{i-1},x_i) \le 3 \cdot 2^{2+i} ct,$$

and so $\ell(\gamma) = \sum_{i=1}^{n+1} \ell(\gamma_i) \le 3 \cdot 2^{4+n} ct$. On the other hand,

$$d(x,y) \ge d(y,p) - d(p,x) \ge 2^n t - t \ge 2^{n-1} t.$$

Consequently, $\ell(\gamma) \le 3 \cdot 2^5 cd(x,y)$.

It remains to show that $\gamma$ satisfies a double cone arc condition. Let $z \in \gamma$, so $z \in \gamma_i$ for some $i$. We consider three cases.

***Case 1.*** $i = 1$.

Now $z \in \gamma[x, x_1]$. As above,

$$\ell(\gamma[x,z]) \le \ell(\gamma[x,x_1]) \le 8cd(x,x_1) \le 24ct,$$

while $\ell(\gamma[z,x_2]) \ge \ell(\gamma[x_1,x_2]) \ge d(x_1,x_2) \ge 2t$. Thus $\ell(\gamma[z,x_2]) \ge \ell(\gamma[x,z])/12c$. Now Lemma 5.11(b) with $K = 4$ applied to $\gamma[x,x_2]$ yields

$$\delta(z) \ge \frac{\ell(\gamma[x,z]) \wedge \ell(\gamma[z,x_2])}{2^5 c(8c+1)^2} \ge \frac{2\ell(\gamma[x,z])}{3 \cdot 2^7 c^2 (8c+1)^2}.$$

***Case 2.*** $2 \le i < n$.

Lemma 5.11(b) with $K = 8$ applied to $\gamma[x_{i-2}, x_{i+1}]$ gives $b_1 \delta(z) \ge \ell(\gamma[x_{i-2},z]) \wedge \ell(\gamma[z,x_{i+1}])$, where $b_1 = 2^7 c(8c+1)^2$. Note that

$$\ell(\gamma[x_{i-2},z]) \ge \ell(\gamma[x_{i-2},x_{i-1}]) \ge d(x_{i-2},x_{i-1}) \ge 2^{i-2} t,$$

and similarly $\ell(\gamma[z,x_{i+1}]) \ge 2^i t$. On the other hand,

$$\ell(\gamma[x,z]) \le \ell(\gamma[x,x_i]) = \sum_{j=1}^{i} \ell(\gamma_j) \le \sum_{j=1}^{i} 3 \cdot 2^{2+j} ct \le 3 \cdot 2^{3+i} ct.$$

It follows that $b_2\delta(z) \ge \ell(\gamma[x,z])$, where $b_2 = 3\cdot 2^5 cb_1 = 3\cdot 2^{12}c^2(8c+1)^2$.

***Case 3.*** $i \in \{n, n+1\}$.

Recall that $d(z,p) \le d(y,p)$ for all $z \in \gamma$ and so $d(z,p) \le 4d(x_{n-1},p)$. Thus Lemma 5.11(b) with $K = 4$ applied to $\gamma[x_{n-1},y]$ gives

$$\ell(\gamma[z,y]) \le \ell(\gamma[x_{n-1},y]) \le 2^4 cd(x_{n-1},y) \le 5\cdot 2^{4+n-1}ct.$$

Since also $\ell(\gamma[x_{n-2},z]) \ge \ell(\gamma[x_{n-2},x_{n-1}]) \ge 2^{n-2}t$, it follows that

$$\ell(\gamma[x_{n-2},z]) \ge \frac{\ell(\gamma[z,y])}{5\cdot 2^5 c}.$$

Applying Lemma 5.11(b) with $K = 8$ to $\gamma[x_{n-2},y]$, we get

$$\delta(z) \ge \frac{[\ell(\gamma[x_{n-2},z]) \wedge \ell(\gamma[z,y])]}{b_1} \ge \frac{\ell(\gamma[z,y])}{5\cdot 2^5 cb_1} = \frac{\ell(\gamma[z,y])}{c_8}. \qquad \square$$

***Remark 5.13.*** The above shows that for $i \le n-1$ and all $z \in \gamma_i$, $\ell(\gamma[x,z]) \le b_2\delta(z)$; we use this fact in the following proof.

5.14. ***Proof of Proposition 5.9.*** First, suppose $d(x,y) \le t/8c$. By Lemma 5.11(b), it suffices to show that $d(z,p) \le 2t$ for $z \in \gamma$. Since $d(x,y) \le t/8c$ and $d(y,p) \ge t$, we see that $d_p(x,y) \le d(x,y)/d(x,p)d(y,p) \le 1/8ct$. If $z \in \gamma$, then (3.1) and the uniformity of $\gamma$ in $I_p(\Omega)$ imply that

$$\frac{d(x,z)}{4td(z,p)} \le d_p(z,x) \le \ell_p(\gamma) \le cd_p(x,y) \le \frac{1}{8t},$$

and so $d(x,z) \le d(z,p)/2$. Now $d(x,p) \ge d(z,p) - d(x,z) \ge d(z,p)/2$, as required.

Next, assume $d(x,y) \ge t/8c$. Set $s = d(y,p)$ and pick $z_0 \in \gamma$ such that $d(z_0,p) \ge d(z,p)$ for all $z \in \gamma$. Set $a = d(z_0,p)$. We claim that $a \le 8cs\rho$. Certainly this is true if $a < 2s$, so suppose $a \ge 2s \ge 2t$. It follows from the triangle inequality that $d(z_0,x)$, $d(z_0,y) \in [a/2, 2a]$ and so

$$\ell_p(\gamma[z_0,x]) \ge d_p(z_0,x) \ge \frac{d(z_0,x)}{4d(z_0,p)d(x,p)} \ge \frac{1}{8t}.$$

Similarly, $\ell_p(\gamma[z_0,y]) \ge 1/8s$, and so the uniformity of $\gamma$ in $I_p(\Omega)$ gives

$$(8cs)^{-1} = [8c(s\vee t)]^{-1} \le \frac{[\ell_p(\gamma[z_0,x]) \wedge \ell_p(\gamma[z_0,y])]}{c} \le \delta_p(z_0).$$

On the other hand, for any $q \in \partial_p\Omega$ we have

$$\delta_p(z_0) \le d_p(z_0,q) \le \frac{d(z_0,q)}{d(z_0,p)d(q,p)} \le \frac{\operatorname{diam}(\Omega)}{a\cdot d(p,q)} = \frac{\operatorname{diam}(\Omega)}{ad(p,q)},$$

and the claim follows by comparing these two estimates for $\delta_p(z_0)$.

Depending on whether the ratios $a/s$ and $a/t$ are less than or exceed 8, we apply either Lemma 5.11(b) (with $K = 8$) or Lemma 5.12 to $\gamma[x, z_0]$ and $\gamma[z_0, y]$ and obtain:

$$\ell(\gamma[x, z_0]) \leq 3 \cdot 2^5 c d(x, z_0) \leq 3 \cdot 2^6 ca,$$

$$\ell(\gamma[y, z_0]) \leq 3 \cdot 2^5 c d(y, z_0) \leq 3 \cdot 2^6 ca.$$

It follows that $\ell(\gamma) \leq 3 \cdot 2^7 ca \leq 3 \cdot 2^{10} c^2 s\rho$. If $s \geq 2t$, then $s/2 \leq d(x, y)$ and so

$$\ell(\gamma) \leq 3 \cdot 2^{11} c^2 \rho d(x, y).$$

If instead $s \leq 2t$, then the assumption $d(x, y) \geq t/8c$ implies that

$$\ell(\gamma) \leq 3 \cdot 2^{11} c^2 t\rho \leq 3 \cdot 2^{14} c^3 \rho d(x, y) = c_4 \rho d(x, y).$$

It remains to show that $\gamma$ satisfies a double cone arc condition. Let $m$, $n$ be the integers such that $2^m t < a \leq 2^{m+1} t$, $2^n s < a \leq 2^{n+1} s$. Let $x_i$, $1 \leq i \leq m$, be the first point on $\gamma[x, z_0]$ (oriented from $x$ to $z_0$) with $d(x_i, p) = 2^i t$ and $y_i$, $1 \leq i \leq n$, be the first point on $\gamma[y, z_0]$ (oriented from $y$ to $z_0$) with $d(y_i, p) = 2^i s$. Let $b_1$, $b_2$ be as in the proof of Lemma 5.12.

Applying Remark 5.13 to $\gamma[x, z]$ and $\gamma[z, y]$, we get that

$$b_2 \delta(z) \geq \ell(\gamma[x, z])$$

for $z \in \gamma[x, x_{m-1}]$ whenever $m \geq 3$, and

$$b_2 \delta(z) \geq \ell(\gamma[z, y])$$

for $z \in \gamma[y_{n-1}, y]$ whenever $n \geq 3$, giving the required double cone arc inequality for all such points.

Eliminate such points from consideration. Set $x' = x_{m-2}$, $x'' = x_{m-1}$ if $m \geq 3$ and $x' = x'' = x$ if $m \leq 2$. Similarly $y' = y_{n-2}$, $y'' = y_{n-1}$ if $n \geq 3$ and $y' = y'' = y$ if $n \leq 2$. We need to get a double cone arc inequality for points $z \in \gamma[x'', y'']$. Applying Lemma 5.11(b) with $K = 8$ to $\gamma[x', y']$, we get

$$b_1 \delta(z) \geq \ell(\gamma[z, x']) \wedge \ell(\gamma[z, y']).$$

Assume that $\ell(\gamma[z, x']) \leq \ell(\gamma[z, y'])$ and so $b_1 \delta(z) \geq \ell(\gamma[z, x'])$; the case $\ell(\gamma[z, x']) \geq \ell(\gamma[z, y'])$ can be handled similarly. If $m \leq 2$, then $x' = x$ and we are done. Suppose instead that $m \geq 3$. Then

$$\begin{aligned}\ell(\gamma[z,x']) = \ell(\gamma[z,x_{m-2}]) &\geq \ell(\gamma[x_{m-1},x_{m-2}]) \\ &\geq d(x_{m-1},x_{m-2}) \geq 2^{m-2}t.\end{aligned}$$

On the other hand,

$$\ell(\gamma[x,z]) \wedge \ell(\gamma[z,y]) \leq \frac{\ell(\gamma)}{2} \leq 3 \cdot 2^6 ca \leq 3 \cdot 2^{m+7} ct.$$

It follows that

$$\begin{aligned}\delta(z) \geq \frac{\ell(\gamma[z,x'])}{b_1} &\geq \frac{[\ell(\gamma[x,z]) \wedge \ell(\gamma[z,y])]}{3 \cdot 2^9 cb_1} \\ &= \frac{[\ell(\gamma[x,z]) \wedge \ell(\gamma[z,y])]}{c_5}.\end{aligned}$$

□

## 6. INVERSIONS AND QUASICONVEXITY

Here we introduce the notion of annular quasiconvexity and demonstrate that a metric space is quasiconvex and annular quasiconvex if and only if its inversions are quasiconvex and annular quasiconvex. These results then produce improved quantitative estimates for the uniformity constants arising in Theorems 5.1 and 5.5.

6.A. ***Annular quasiconvexity.*** Given $c \geq 1$, we call a metric space $X$ *$c$-annular quasiconvex at $p \in X$* provided for all $r > 0$, points in $A(p;r,2r)$ can be joined by $c$-quasiconvex paths lying in $A(p;r/c,2cr)$. We call $X$ *$c$-annular quasiconvex* if it is $c$-annular quasiconvex at each point. Examples of quasiconvex and annular quasiconvex metric spaces include Banach spaces and upper regular Loewner spaces; the latter includes Carnot groups and certain Riemannian manifolds with non-negative Ricci curvature; see [9, 3.13, 3.18, Section 6]. Korte [12] has recently verified that doubling metric measure spaces which support a $(1,p)$-Poincaré inequality with sufficiently small $p$ are annular quasiconvex.

Here is a 'bootstrapping' technique which we find useful.

***Lemma 6.1.*** *Let $(X,d)$ be $c$-quasiconvex and $c$-annular quasiconvex at $p \in X$. Fix $0 < 2r \leq R$. Points in $A(p;r,R)$ can be joined by $5c$-quasiconvex paths which stay in $A(p;r/c,cR)$.*

*Proof.* Let $x$, $y \in A(p;r,R)$. Put $d(x,p) = r$, $d(y,p) = R$ and suppose $R > 4r$. Pick any $c$-quasiconvex path $\eta$ joining $x$, $y$. Let $\tilde{\eta}$ be a component of $\eta \cap A(p;2r,R/2)$ with endpoints $u$, $v$ satisfying $d(u,x) = 2r$ and $d(v,x) = R/2$. Use annular quasiconvexity to choose $c$-quasiconvex paths $\alpha$ and $\beta$ joining $x$, $u$ in $A(p;r/c,2cr)$ and $y$, $v$ in $A(x;R/2c,cR)$, respectively. The concatenation

$\gamma$ of the three paths $\alpha$, $\tilde{\eta}$, $\beta$ has the required properties. In fact, $d(x,y) \geq (3R/4) \vee (3r)$, and therefore

$$\ell(\gamma) = \ell(\alpha) + \ell(\tilde{\eta}) + \ell(\beta) \leq c\left(3r + d(x,y) + \frac{3R}{2}\right) \leq 4cd(x,y).$$

If instead $r \leq R/2 \leq 2r$, we argue as above but replace $\tilde{\eta}$ by $\{z\}$ for any point $z$ satisfying $d(z,p) = R/2$, and then $d(x,y) \geq (R/2) \vee r$, so the concatenated path $\gamma$ satisfies

$$\ell(\gamma) \leq \ell(\alpha) + \ell(\beta) \leq c\left(r + \frac{R}{2}\right) + c\left(\frac{3R}{2}\right) \leq 5cd(x,y). \qquad \square$$

According to Proposition 4.2, local quasiconvexity is preserved under both inversion and sphericalization. It is easy to see that global quasiconvexity does not share this property. Indeed, $\mathrm{Inv}_p(X)$ may fail to be quasiconvex even when $(X,d)$ is a bounded length space.

***Example 6.2.*** Let $X$ be the subset of $\mathbb{C}$ given as

$$X = [0,1] \cup \bigcup_{n=1}^{\infty} [a_n, a_n + ib_n]$$

where $1 \geq a_1 > a_2 > \cdots > a_n \to 0$ and $0 < b_n \leq 1$. We equip $X$ with its Euclidean length distance which we denote by $d$. Then $(X,d)$ is a length space of diameter at most 3. Let $p = 0$ and $c_n = a_n + ib_n$. Choosing $a_n = 1/2n^2$ and $b_n = 1/n$ we find that

$$d_p(c_n, c_{n+1}) \leq i_p(c_n, c_{n+1}) = \frac{b_n + a_n - a_{n+1} + b_{n+1}}{(a_n + b_n)(a_{n+1} + b_{n+1})} \leq 3(n+1).$$

On the other hand, $d_p(c_n, a_n) \geq i_p(c_n, a_n)/4 \geq n^2/16$. Since any path from $c_n$ to $c_{n+1}$ must pass through $a_n$, $l_p(c_n, c_{n+1}) \geq n^2/16$. Thus $d_p$ and $l_p$ are not bilipschitz equivalent.

***Proposition 6.3.*** *Suppose $X$ is connected and $c$-annular quasiconvex at some point $p \in X$. Then $X$ is $9c$-quasiconvex and* $\mathrm{Inv}_p(X)$ *is $72c^3$-quasiconvex.*

*Proof.* We establish the latter assertion and leave the former for the interested reader. Let $x, y \in X_p$, $t = d(x,p)$, $s = d(y,p)$ and assume $t \leq s$. Suppose $s \leq 2t$. Then $y \in A(p;t,2t)$, so there is a $c$-quasiconvex $\gamma$ joining $x$, $y$ in $A(p;t/c,2ct)$. According to Lemma 5.6(a), $\gamma$ is $8c^3$-quasiconvex in $\mathrm{Inv}_p(X)$.

Suppose instead that $s > 2t$. Let $n \geq 2$ be the integer with $2^{n-1}t < d(y,p) \leq 2^n t$. Put $x_0 = x$, $x_n = y$ and for $1 \leq i < n$ select points $x_i$ with $d(x_i,p) = 2^i t$. For $0 \leq i < n$, annular quasiconvexity provides $c$-quasiconvex paths $\gamma_i \subset A(p; 2^i t/c, c2^{i+1}t)$ joining $x_i$ to $x_{i+1}$. Note that $d(x_i, x_{i+1}) \le 3 \cdot 2^i t$ and thus $\ell_p(\gamma_i) \le 3c^3/(2^i t)$ by (5.2). Letting $\gamma$ be the concatenation of $\gamma_0, \dots, \gamma_{n-1}$ we have

$$\ell_p(\gamma) = \sum_{i=0}^{n-1} \ell_p(\gamma_i) \le \frac{3c^3}{t} \sum_{i=0}^{n-1} 2^{-i} = \frac{3c^3}{t}[2 - 2^{1-n}].$$

On the other hand,

$$d_p(x, y) \ge \frac{d(x, y)}{4d(x, p)d(y, p)} \ge \frac{s - t}{4ts} \ge \frac{(2^{n-1} - 1)t}{4 \cdot 2^n t^2} = \frac{1}{8t}[1 - 2^{1-n}].$$

It now follows that $\ell_p(\gamma) \le 72c^3 d_p(x, y)$.

The case $x \in X_p$ and $y = p'$ (and $X$ unbounded) is handled in the same manner as the $s > 2t$ case, except that the lower bound for $d_p(x, y)$ is now $d_p(x, p') \ge 1/4t$. ◻

6.B. ***Invariance of quasiconvexity.*** Here are our main invariance results for quasiconvexity. In each of 6.4(a,b) and 6.5(a,b), $c_1 = 72c^3$ and $c_2 = 3912c^5$.

***Theorem 6.4.*** *Let $(X, d)$ be a metric space and $p \in X$.*

(a) *If $(X, d)$ is $c$-quasiconvex and $c$-annular quasiconvex, then $(\mathrm{Inv}_p(X), d_p)$ is $c_1$-quasiconvex and $c_2$-annular quasiconvex;*
(b) *If $p$ is a non-isolated point in $(X, d)$ and $(\mathrm{Inv}_p(X), d_p)$ is $c$-quasiconvex and $c$-annular quasiconvex, then $(X, d)$ is $16^2 c_1$-quasiconvex and $16^2 c_2$-annular quasiconvex.*

***Theorem 6.5.*** *Let $(X, d)$ be an unbounded metric space and $p \in X$.*

(a) *If $(X, d)$ is $c$-quasiconvex and $c$-annular quasiconvex, then $(\mathrm{Sph}_p(X), \hat{d}_p)$ is $c_1$-quasiconvex and $c_2$-annular quasiconvex.*
(b) *If $(\mathrm{Sph}_p(X), \hat{d}_p)$ is $c$-quasiconvex and $c$-annular quasiconvex, then $(X, d)$ is $2^8 c_1$-quasiconvex and $2^8 c_2$-annular quasiconvex.*

We break the proofs into a number of pieces as indicated below. Establishing 6.4(a) turns out to be the crucial step; then we appeal to results from Subsection 3.C.

*Proof of Theorem* 6.4(a). We assume $X$ is both $c$-quasiconvex and $c$-annular quasiconvex.

Thanks to Proposition 6.3 we already know that $\mathrm{Inv}_p(X)$ is $c_1$-quasiconvex, so it remains to demonstrate the annular quasiconvexity. Let $x \in X_p$, $r > 0$, and set $t = d(x, p)$. Note that $rt \le d(x, y)/d(y, p) \le 8rt$ for $y \in A_p(x; r, 2r)$. We consider three cases.

***Case 1.*** $rt \le 1/80c$.

We first claim that

$$A_p(x;r,2r) \subset A\left(x;\ \frac{10rt^2}{11},\ \frac{80rt^2}{9}\right).$$

To see this, let $y \in A_p(x;r,2r)$. Combining the inequality $|d(y,p) - t| \le d(x,y)$ with the estimate

$$\frac{d(x,y)}{d(y,p)} \le \frac{1}{10c} \le \frac{1}{10},$$

we see that $10t/11 \le d(y,p) \le 10t/9$, and so $10rt^2/11 \le d(x,y) \le 80rt^2/9$, as claimed.

Let $y_1$, $y_2 \in A_p(x;r,2r)$. By our claim and Lemma 6.1, there is a path $\gamma$ from $y_1$ to $y_2$ with $\ell(\gamma) \le 5cd(y_1,y_2)$ and $\gamma \subset A(x;10rt^2/11c, 80crt^2/9)$. We show that $\gamma$ also satisfies annular quasiconvexity conditions with respect to $d_p$.

Since $rt \le 1/80c$, we have $d(x,z) \le 80crt^2/9 \le t/9$, for all $z \in \gamma$. By the triangle inequality, $8t/9 \le d(p,z) \le 10t/9$ for all $z \in \gamma$, and so

$$\ell_p(\gamma) \le \frac{\ell(\gamma)}{(8t/9)^2} \le \frac{81 \cdot 5cd(y_1,y_2)}{64t^2}.$$

On the other hand,

$$d_p(y_1,y_2) \ge \frac{d(y_1,y_2)}{4d(y_1,p)d(y_2,p)} \ge \frac{d(y_1,y_2)}{4(10t/9)^2}.$$

It follows that $\ell_p(\gamma) \le 125cd_p(y_1,y_2)/4$.

It remains to prove that $\gamma$ is contained in a $d_p$-annulus. Let $z \in \gamma$. An upper bound on $d_p(z,x)$ is easy to determine: since $y_1$, $y_2 \in A_p(x;r,2r)$ and $\ell_p(\gamma) \le 125cd_p(y_1,y_2)/4 \le 125cr$, the triangle inequality gives $d_p(z,x) \le 2r + \ell_p(\gamma)/2 \le 129cr/2$. As for a lower bound,

$$\begin{aligned} d_p(z,x) &\ge \frac{d(z,x)}{4d(z,p)d(x,p)} \ge \frac{d(z,x)}{4(10t^2/9)} \\ &= \frac{9}{40t^2} d(z,x) \ge \frac{9}{40t^2} \cdot \frac{10rt^2}{11c} = \frac{9r}{44c}. \end{aligned}$$

***Case 2.*** $rt > 10c$.

We first claim that $A_p(x;r,2r) \subset A(p;5/44r,10/9r)$. To see this, let $y \in A_p(x;r,2r)$. Combining the inequality $|d(x,y) - t| \le d(y,p)$ with the estimate $d(x,y)/d(y,p) > 10c \ge 10$, we see that $10t/11 \le d(x,y) \le 10t/9$.

Since $rt \le d(x,y)/d(y,p) \le 8rt$, we have $d(y,p) \le d(x,y)/rt \le 10/9r$ and $d(y,p) \ge d(x,y)/8rt \ge 5/44r$.

Let $y_1, y_2 \in A_p(x;r,2r)$. By our claim and Lemma 6.1, there is a path $\gamma$ from $y_1$ to $y_2$ with $\ell(\gamma) \le 5cd(y_1,y_2)$ and $\gamma \subset A(p;5/44cr,10c/9r)$. We show that $\gamma$ also satisfies annular quasiconvexity conditions with respect to $d_p$.

We have

$$\ell_p(\gamma) \le \left(\frac{5}{44cr}\right)^{-2} \ell(\gamma) \le \left(\frac{44cr}{5}\right)^2 5cd(y_1,y_2).$$

On the other hand,

$$d_p(y_1,y_2) \ge \frac{d(y_1,y_2)}{4d(y_1,p)d(y_2,p)} \ge \frac{d(y_1,y_2)}{4(10c/9r)^2} = \frac{81r^2 d(y_1,y_2)}{2^4 5^2 c^2}.$$

It follows that $\ell_p(\gamma) \le 1936c^5 d_p(y_1,y_2)$.

It remains to prove that $\gamma$ is contained in a $d_p$-annulus. Let $z \in \gamma$. As in Case 1, an upper bound on $d_p(z,x)$ is easy: we get that $d_p(z,x) \le 2r + 1936c^5 \cdot 2r < c_2 r$. As for the lower bound, since $z \in \gamma \subset A(p;5/44cr,10c/9r)$, we have *a fortiori* $d(z,p) \le 2c/r$. But by assumption $d(x,p) > 10c/r$, and so $d(x,z) \ge d(x,p)/2$. Thus

$$d_p(z,x) \ge \frac{d(z,x)}{4d(z,p)d(x,p)} \ge \frac{d(x,p)/2}{8c\, d(x,p)/r} = \frac{r}{16c}.$$

***Case 3.*** $1/80c \le rt \le 10c$.

Let $y_1, y_2 \in A_p(x;r,2r)$. By quasiconvexity, there is a path $\gamma$ from $y_1$ to $y_2$ with $\ell_p(\gamma) \le c_1 d_p(y_1,y_2)$. Set $r' = r/800c^2$. If $\gamma \cap B_p(x,r') = \emptyset$, then we are done, so assume $\gamma \cap B_p(x,r') \ne \emptyset$. Let $z_1$, $z_2$ be the first and last points on $\gamma$ with $d_p(x,z_i) = r'$. By our assumption, we have $r't \le 1/80c$. By Case 1, there is a path $\gamma'$ from $z_1$ to $z_2$ such that $\ell_p(\gamma') \le 125cd_p(z_1,z_2)/4$ and $\gamma' \subset A_p(x;9r'/44c,129cr'/2)$. We now have

$$\frac{4\ell_p(\gamma')}{125c} \le d_p(z_1,z_2) \le \ell_p(\gamma[z_1,z_2]) \le \ell_p(\gamma) \le c_1 d_p(y_1,y_2).$$

Replace the part $\gamma[z_1,z_2]$ of $\gamma$ by $\gamma'$ to obtain a new path $\gamma''$ from $y_1$ to $y_2$. Note that $\gamma''$ is disjoint from $B_p(x,9r'/44c) \supset B_p(x,r/3912c^3)$ and that

$$\ell_p(\gamma'') \le c_1\left(1+\frac{125c}{4}\right) d_p(y_1,y_2) \le 2322c^4 d_p(y_1,y_2).$$

It follows that $\gamma'' \subset A_p(x;r/3912c^3,4646c^4 r)$.

It remains to consider the case when $X$ is unbounded and $x = p'$. Let $r > 0$. From the definition of $d_p(y,p')$, we see that $A_p(p';r,2r) \subset A(p;1/8r,1/r)$.

Let $y_1, y_2 \in A_p(p'; r, 2r)$. Then there is a path $\gamma$ from $y_1$ to $y_2$ with $\ell(\gamma) \le 5cd(y_1, y_2)$ and $\gamma \subset A(p; 1/8cr, c/r)$, so

$$\ell_p(\gamma) \le \frac{\ell(\gamma)}{(1/8cr)^2} \le (8cr)^2 5cd(y_1, y_2) = 320c^3 r^2 d(y_1, y_2)$$

and

$$d_p(y_1, y_2) \ge \frac{d(y_1, y_2)}{4d(y_1, p)d(y_2, p)} \ge \frac{d(y_1, y_2)}{4(1/r)^2} = \frac{r^2 d(y_1, y_2)}{4}.$$

It follows that $\ell_p(\gamma) \le 1280c^3 d_p(y_1, y_2)$. Let $z \in \gamma$. Then $d_p(z, p') \le 2562c^3 r$, and $d_p(z, p') \ge 1/4d(p, z) \ge r/4c$, as required. □

*Proof of Theorem* 6.5(a). We proceed as in the proof of Theorem 6.4(a). □

*Proof of Theorem* 6.4(b). Assume $p \in X$ is non-isolated and $(\mathrm{Inv}_p(X), d_p)$ is $c$-quasiconvex and $c$-annular quasiconvex. First, suppose $X$ is unbounded. As in Proposition 3.3, letting $d' = (d_p)_{p'}$ denote the distance on $\mathrm{Inv}_{p'}(\mathrm{Inv}_p X)$ we have $(X, d)$ 16-bilipschitz equivalent to $(X, d')$. Appealing to Theorem 6.4(a) we find that $(X, d')$ is $c_1$-quasiconvex and $c_2$-annular quasiconvex, so our asserted conclusion follows.

Next, suppose $X$ is bounded. Let $r = \mathrm{diam}(X)$ and fix a point $q \in X$ with $d(p, q) \ge r/2$. Since both quasiconvexity and annular quasiconvexity are preserved by dilations (with no change in the associated constants), we can rescale our distance and we find that

$$(\mathrm{Inv}_p(X), (d/r)_p) = (\mathrm{Inv}_p(X), r\, d_p)$$

is both $c$-quasiconvex and $c$-annular quasiconvex. Set $(Y, e) = (\mathrm{Inv}_p(X), (d/r)_p)$. By Theorem 6.5(a), $(\mathrm{Sph}_q(Y), \hat{e}_q)$ is $c_1$-quasiconvex and $c_2$-annular quasiconvex. Proposition 3.5 tells us that $(X, d/r)$ and $(\mathrm{Sph}_q(Y), \hat{e}_q)$ are 256-bilipschitz equivalent. It now follows that $(X, d/r)$ is $2^{16}c_1$-quasiconvex and $2^{16}c_2$-annular quasiconvex, so $(X, d)$ is also. □

*Proof of Theorem* 6.5(b). Here we proceed as in the unbounded case of the proof of Theorem 6.4(b), only now we employ Proposition 3.4. □

6.C. ***Connection with uniformity.*** As an application of our results in Subsection 6.B, we prove Theorems 6.7 and 6.8 below which, in contrast with Theorems 5.1 and 5.5, provide better parameter dependence when our ambient space is quasiconvex and annular quasiconvex.

First we point out that uniformity does not imply annular quasiconvexity; e.g., $(-1, 1)$ is a uniform subspace of $\mathbb{R}$, but it is not annular quasiconvex at the origin. On the other hand, it is not hard to see that if $\zeta$ is a non-isolated point of some $\Omega \subset X$ and $\Omega \setminus \{\zeta\}$ is $b$-uniform, then $\Omega$ is both $b$-uniform and $2b$-annular quasiconvex at $\zeta$. (The uniformity of $\Omega$ follows from the fact that $\zeta \in \partial(\Omega \setminus \{\zeta\})$ and we can always join any two points in the closure of a uniform space with a uniform arc as explained at the end of Subsection 5.A).

Here is a sort of converse to these remarks.

***Theorem 6.6.*** *Let $X$ be a complete metric space. Suppose $\Omega \subset X$ is a locally compact $b$-uniform subspace (so open with $\partial\Omega \neq \emptyset$). Let $\zeta \in \Omega$ and put $\Omega' = \Omega \setminus \{\zeta\}$. If $X$ is $c$-annular quasiconvex at $\zeta$, then $\Omega'$ is $b'$-uniform with $b' = (5bc)^2$.*

*Proof.* Put $a = 5bc$. For $z \in \Omega'$, write $\delta'(z) = \mathrm{d}(z, \partial\Omega') = \delta(z) \wedge d(z,\zeta)$. Fix $x, y \in \Omega'$. Assume $t = d(x,\zeta) \le d(y,\zeta)$. Let $\gamma$ be a quasihyperbolic geodesic in $\Omega$ from $x$ to $y$. Then $\gamma$ is a $b$-uniform arc in $\Omega$ (as are all its subarcs). In particular, $\gamma$ is $b$-quasiconvex, so if $\gamma$ satisfies an appropriate double cone condition in $\Omega'$, we are done. Therefore we may assume there is some $z \in \gamma$ with

$$2ab\delta'(z) \le \ell(\gamma[x,z]) \wedge \ell(\gamma[y,z]) \le bd(x,z).$$

Since $\gamma$ is a double $b$-cone in $\Omega$, we must have

$$d(z,\zeta) = \delta'(z) \le \frac{d(z,x)}{2a};$$

so $(2a-1)d(z,\zeta) \le t$, which implies $d(z,\zeta) < t/a$.

In particular, we may assume $\gamma \cap B(\zeta; t/a) \neq \emptyset$. Now let $x'$ and $y'$ be the first and last points respectively on $\gamma$ with $d(x',\zeta) = d(y',\zeta) = t/a$. Put $\gamma_x = \gamma[x,x']$ and $\gamma_y = \gamma[y,y']$. Since $X$ is $c$-annular quasiconvex at $\zeta$, there is a $c$-quasiconvex path $\alpha \subset A(\zeta; t/ac, 2ct/a)$ joining $x'$ and $y'$. Set $\beta = \gamma_x \cup \alpha \cup \gamma_y$. We verify that $\beta$ is a $b'$-uniform curve in $\Omega'$.

Notice that $\ell(\alpha) \le cd(x',y') \le 2ct/a$ and also $\ell(\alpha) \le c\ell(\gamma[x',y'])$. It follows that

$$\ell(\beta) = \ell(\gamma_x) + \ell(\alpha) + \ell(\gamma_y) \le c\ell(\gamma) \le bcd(x,y),$$

so in fact $\beta$ is $bc$-quasiconvex. Note that similar reasoning provides the estimates

$$(6.1)\quad \forall\, z \in \gamma_x \cup \gamma_y: \quad \ell(\beta[x,z]) \wedge \ell(\beta[y,z]) \le c[\ell(\gamma[x,z]) \wedge \ell(\gamma[y,z])].$$

It remains to prove the double cone arc condition. We first consider points $z \in \alpha$; so $t/ac \le d(z,\zeta) \le 2ct/a$. Now

$$d(x',z) \le \ell(\alpha) \le \frac{2ct}{a} \quad \text{and} \quad b\delta(x') \ge \ell(\gamma_x) \wedge \ell(\gamma_y) \ge t - \frac{t}{a},$$

so

$$\delta(z) \ge \delta(x') - d(x',z) \ge [(a-1) - 2bc]\left(\frac{t}{ab}\right) \ge \frac{2ct}{a} \ge d(z,\zeta);$$

here the penultimate inequality holds by our choice of $a$. It follows that $\delta'(z) = d(z,\zeta) \ge t/ac$ and thus

$$\ell(\beta[x,z]) \wedge \ell(\beta[y,z]) \le \ell(\alpha) + \ell(\gamma_x) \le [2c + b(a+1)]\left(\frac{t}{a}\right) \le b'\delta'(z)$$

as desired.

Now suppose $z \in \gamma_x \cup \gamma_y$. Then $d(z,\zeta) \ge t/a$ by our choice of $x'$ and $y'$. We claim that

$$\ell(\beta[x,z]) \wedge \ell(\beta[y,z]) \le a^2\delta'(z).$$

Suppose this were false. Then, using (6.1) in conjunction with $\gamma$ being a double $b$-cone arc in $\Omega$, we would obtain $a^2\delta'(z) < bc\delta(z)$ which in turn would imply that

$$\begin{aligned} d(z,\zeta) = \delta'(z) &\le \frac{c}{a^2}[\ell(\gamma[x,z]) \wedge \ell(\gamma[y,z])] \\ &\le \frac{c}{a^2}\ell(\gamma[x,z]) \le \frac{bc}{a^2}d(z,x) < \frac{d(z,x)}{2a}; \end{aligned}$$

but as in the beginning of the proof, this would give the contradiction $d(z,\zeta) < t/a$. □

Finally, here are our improved versions of Theorems 5.1 and 5.5.

***Theorem 6.7.*** *Let $X$ be a complete $c$-quasiconvex $c$-annular quasiconvex metric space and fix $p \in X$. Suppose $\Omega \subset X_p$ is open and locally compact with $\partial\Omega \ne \emptyset \ne \partial_p\Omega$. Then $\Omega$ is uniform if and only if $I_p(\Omega)$ is uniform. The uniformity constants depend only on each other and $c$.*

*Proof.* Since $r(p) \le c$, most of this follows already from Theorem 5.1(a,b). It suffices to consider the case when $\Omega$ is bounded and $I_p(\Omega)$ is $b$-uniform. Fix any point $q \in \partial\Omega$ and pick $\zeta \in \Omega$ with $d(q,\zeta) > \operatorname{diam}(\Omega)/3$. Set $\Omega' = \Omega \setminus \{\zeta\}$. Then $\partial\Omega' = \partial\Omega \cup \{\zeta\}$ and $\operatorname{diam}(\partial\Omega') > \operatorname{diam}(\Omega)/3$.

According to Theorem 6.4(a), $\operatorname{Inv}_p(X)$ is $c_1$-quasiconvex and $c_2$-annular quasiconvex. Since $I_p(\Omega)$ is $b$-uniform, Theorem 6.6 says that $I_p(\Omega') = I_p(\Omega) \setminus \{\zeta\}$ is $b'$-uniform with $b' = b'(b,c) = (5bc_2)^2$. Theorem 5.1(c) asserts that $\Omega'$ is $b''$-uniform, which then implies that $\Omega$ is $b''$-uniform. Here $b'' = c_0 \operatorname{diam}(\Omega')/b(p)$ and $c_0 = c_0(b') = c_0(b,c)$. Recalling that $b(p) \ge \operatorname{diam}(\partial\Omega')/2 \ge \operatorname{diam}(\Omega)/6$, we find that $b'' \le 6c_0$. □

***Theorem 6.8.*** *Let $X$ be an unbounded complete $c$-quasiconvex $c$-annular quasiconvex metric space and fix $p \in X$. Suppose $\Omega \subset X$ is open and locally compact with $\partial\Omega \ne \emptyset$. Then $\Omega$ is uniform if and only if $S_p(\Omega)$ is uniform. The uniformity constants depend only on each other and $c$.*

*Proof.* In view of Theorem 5.5(b), we only need to show that $S_p(\Omega)$ uniform implies $\Omega$ is too. As in the proof of Theorem 5.5(a), we use Proposition 3.4 which says that the identity map $(X,d) \to (\mathrm{Inv}_{\hat{p}}(Y), d') = (X, d')$ is 16-bilipschitz, where $Y = \mathrm{Sph}_p(X)$ and $d' = (\hat{d}_p)_{\hat{p}}$. The result now follows from Theorem 6.7. ◻

## 7. GENERALIZED INVERSION

In [1, Section 3] Balogh and the first author investigated the notion of *flattening* wherein a closed subset of the metric boundary of a suitable incomplete bounded metric space is sent to infinity in a manner similar to the way $\mathrm{Inv}_p$ sends a point $p$ to infinity. Let $X$ be complete, $p \in X$, and suppose $X_p$ is such a space. If we flatten $X_p$ using the *standard flattening function* $t \mapsto t^{-2}$, then—recalling (4.1)—we obtain the length distance $l_p$ associated with $d_p$. Thus the standard flattened metric on $X_p$ is bilipschitz equivalent to $d_p$ precisely when $\mathrm{Inv}_p(X)$ is quasiconvex (which, according to Proposition 6.3, is true whenever $X$ is annular quasiconvex).

The notion of flattening allows for more general flattening functions. Inspired by this, we consider *generalized inversion* defined for points $x, y \in X_p$ by

$$i_{p,f}(x,y) := [d(x,y) \wedge d(x,p) \wedge d(y,p)] \cdot f(d(x,p) \wedge d(y,p)),$$

$$d_{p,f}(x,y) := \inf\left\{ \sum_{i=1}^{k} i_{p,f}(x_i, x_{i-1}) : x = x_0, \dots, x_k = y \in X_p \right\};$$

here $I \xrightarrow{f} (0,\infty)$ is a continuous function satisfying

(I-0) $f(r) \le Cf(s)$ when $\frac{1}{2} \le r/s \le 2$ and $r, s \in I$,

and the function $F(t) := tf(t)$ satisfies

(I-1) $F(s) \le CF(r)$ when $r \le s$ and $r, s \in I$,

(I-2) $F(r) \to \infty$ as $r \to 0$, and

(I-3) when $X$ is unbounded, $F(r) \to 0$ as $r \to \infty$.

In the above, $C > 2$ is a constant and either $I = (0,\infty)$ if $X$ is unbounded or $I = (0, \operatorname{diam} X]$. We call any $f$ which satisfies (I-0)–(I-3) a *C-admissable inversion function*.

We will see that for all points $x, y \in X_p$,

$$C^{-2} i_{p,f}(x,y) \le d_{p,f}(x,y) \le i_{p,f}(x,y) \le F(d(x,p)) \vee F(d(y,p)) \tag{7.1}$$

and so $d_{p,f}$ is a distance function which can be extended in the usual way to $\hat{X}_p$. Thus we obtain a metric space $(\mathrm{Inv}_{p,f}(X), d_{p,f})$ (and when $X$ is unbounded we include a point $p'$ in $\mathrm{Inv}_{p,f}(X)$ which corresponds to the point at infinity). Before proceeding, we discuss this definition. First, it is reasonable to call this process generalized inversion because $i_{p,f}$ is comparable to $i_p$ in the case of the standard (4-admissable) inversion function $t \mapsto t^{-2}$. To see this, suppose $x, y \in X_p$, with

$d(x,p) \le d(y,p)$. If $d(x,y) \le 2d(x,p)$, then $d(y,p) \le d(y,x)+d(x,p) \le 3d(x,p)$, and so

$$i_p(x,y) = \frac{d(x,y)}{d(x,p)d(y,p)} \le \frac{d(x,y)}{d(x,p)^2} \le 2i_{p,f}(x,y) \le 6i_p(x,y).$$

On the other hand, if $d(x,y) > 2d(x,p)$, then $|d(y,p)-d(y,x)| \le d(x,p) < d(x,y)/2$, so $\frac{2}{3} < d(x,y)/d(y,p) < 2$ and therefore

$$\frac{1}{2}i_p(x,y) < \frac{1}{d(x,p)} = i_{p,f}(x,y) < \frac{3}{2}i_p(x,y).$$

Next, our definition of admissable inversion functions is quite natural. In particular, (I-1) ensures that inversion dilates distances close to $p$ more than those far from $p$, (I-2) ensures that distances blow up near $p$, and (I-3) ensures that all sequences in $X$ that tend to $\infty$ in $\hat{X}$ tend to a unique point $p'$ in the completion of $(X_p, d_{p,f})$. Also, assuming (I-1), (I-0) is equivalent to the fact that $\int_r^{2r} f(t)\,\mathrm{d}t$ is comparable to $F(r)$. To see why this is needed, let us perform our generalized inversion on the Euclidean half-line $[0,\infty)$ with $p=0$. We find that $i_{p,f}(r,2r) = rf(r) = F(r)$ and $d_{p,f}(r,2r) \le \int_r^{2r} f(t)\,\mathrm{d}t$. If the latter were much smaller than $F(r)$, we would lose the basic property of our inversion theory that says $i_{p,f}$ should be comparable with $d_{p,f}$.

We note that all flattening functions are admissable inversion functions; this follows easily from [1, Lemma 3.3]. Admissable inversion functions form a strictly larger class than flattening functions since they may decay at a slower rate: for instance, $f(t) = t^{-1}[\log(1+1/t)]^\alpha$ is an admissable inversion function for all $\alpha > 0$. Note however that functions with exponential decay such as $f(t) = t^{-2}\exp(-\alpha t)$ are not admissable because they violate (I-0).

Now we examine some of our earlier results to see their generalized versions. The following is an analogue of Lemma 3.1(a,c); part (b) of that lemma does not generalize.

***Lemma 7.1.*** *Let $(X,d)$ be a metric space with fixed base point $p \in X$, and let $f$ be a $C$-admissable inversion function for some $C > 2$.*

(a) *The inequalities in (7.1) hold for all $x,\ y \in \mathrm{Inv}_{p,f}(X)$ and $d_{p,f}$ is a distance on $\mathrm{Inv}_{p,f}(X)$.*

(b) *$(\mathrm{Inv}_{p,f}(X), d_{p,f})$ is bounded if and only if $p$ is an isolated point in $(X,d)$ in which case*

$$C^{-2}\left(F(\delta) \wedge \left[\frac{\operatorname{diam}(X_p) f(\delta)}{2}\right]\right) \le \operatorname{diam}_{p,f}(\hat{X}_p) \le CF(\delta),$$

*where $\delta = \mathrm{d}(p, X_p) > 0$, and $\operatorname{diam}_{p,f}$ denotes diameter with respect to the metric $d_{p,f}$.*

*Proof.* It suffices to verify the inequalities in (7.1) for $x$, $y \in X_p$, for if $X$ is unbounded and one of these points happens to be $p'$, then we simply look at the appropriate limit. The right hand inequalities there follow from the definitions of $d_{p,f}$ and $i_{p,f}$. In fact if $d(x,p) \le d(y,p)$, then $d_{p,f}(x,y) \le i_{p,f}(x,y) \le F(d(x,p))$.

We assume $d(x,p) \le d(y,p)$ and prove the leftmost inequality. Thus either $d(x,y) \le d(x,p)$, in which case $i_{p,f}(x,y) = d(x,y)f(d(x,p))$, or $d(x,y) > d(x,p)$ and $i_{p,f}(x,y) = F(d(x,p))$. Let $x_0,\dots,x_k$ be an arbitrary sequence of points in $X_p$ with $x_0 = x$ and $x_k = y$. We consider two main cases. If $d(x_i,p) \le 2d(x,p)$ for all $i$, then it follows that

$$\begin{aligned}\sum_{i=1}^{k} i_{p,f}(x_i,x_{i-1}) &\ge C^{-1}\sum_{i=1}^{k}[d(x_i,x_{i-1}) \wedge d(x,p)]f(d(x,p))\\ &\ge C^{-1}[d(x,y) \wedge d(x,p)]f(d(x,p))\\ &= C^{-1} i_{p,f}(x,y).\end{aligned}$$

Note that the first inequality (inside the summation sign) follows by either (I-1) or (I-0) depending on whether or not $d(x_{i-1},p) \wedge d(x_i,p) \le d(x,p)$.

Suppose instead that there exists some $j \in \{1,\dots,k\}$ such that $d(x_j,p) > 2d(x,p)$; we choose the smallest such $j$. We consider two subcases. First, assume some $j' \in \{1,\dots,j\}$ exists so that

$$\frac{d(x_{j'-1},p)}{d(x_{j'},p)} \notin [\tfrac{1}{2},2].$$

If $d(x_{j'-1},p)/d(x_{j'},p) < \frac{1}{2}$, define $u = x_{j'-1}$ and $v = x_{j'}$; otherwise switch the definitions of $u$, $v$. Then $d(u,p) \le 2d(x,p)$ and so again by either (I-0) or (I-1),

$$\begin{aligned}\sum_{i=1}^{j'} i_{p,f}(x_i,x_{i-1}) &\ge i_{p,f}(u,v) = F(d(u,p))\\ &\ge C^{-1}F(d(x,p)) \ge C^{-1} i_{p,f}(x,y).\end{aligned}$$

Next, assume $d(x_{i-1},p)/d(x_i,p) \in [\frac{1}{2},2]$ for all $i \in \{1,\dots,j\}$. Let $G(t) = \int_t^D f(s)\,\mathrm{d}s$ for $0 < t < D = \operatorname{diam}(X)$. We claim that if $u$, $v \in X_p$ and $d(u,p)/d(v,p) \in [\frac{1}{2},2]$, then

$$|G(d(u,p)) - G(d(v,p))| \le C i_{p,f}(u,v).$$

Assuming this claim for the moment, we see by the triangle inequality and (I-0) that

$$
\begin{aligned}
\sum_{i=1}^{k} i_{p,f}(x_i, x_{i-1}) &\ge \sum_{i=1}^{j} i_{p,f}(x_i, x_{i-1}) \\
&\ge C^{-1} \sum_{i=1}^{j} |G(d(x_i, p)) - G(d(x_{i-1}, p))| \\
&\ge C^{-1}[G(d(x,p)) - G(2d(x,p))] \\
&\ge C^{-2} F(d(x,p)) \ge C^{-2} i_{p,f}(x,y).
\end{aligned}
$$

To justify the claim, we may assume $d(u,p) \le d(v,p)$. By (I-0) we see that

$$0 \le G(d(u,p)) - G(d(v,p)) \le C[d(v,p) - d(u,p)]f(d(u,p)).$$

But

$$d(v,p) - d(u,p) = [d(v,p) - d(u,p)] \wedge d(u,p) \le d(u,v) \wedge d(u,p).$$

Substituting this last estimate into the previous one substantiates the claim.

It is easy to deduce the first assertion in (b) from (I-2). Suppose $\delta = \mathrm{d}(p, X_p) > 0$. The upper bound $\operatorname{diam}_{p,f} \hat{X}_p \le CF(\delta)$ follows from (7.1) along with (I-1). For the lower bound, let $\varepsilon > 0$ and select points $a_\varepsilon, b_\varepsilon \in X_p$ with $d(a_\varepsilon, p) \le \delta + \varepsilon$, $d(b_\varepsilon, p) \ge d(a_\varepsilon, p)$, and $d(a_\varepsilon, b_\varepsilon) \ge (\operatorname{diam}(X_p) - \varepsilon)/2$. Using part (a), we see that

$$
\begin{aligned}
\operatorname{diam}_{p,f} \hat{X}_p &\ge d_{p,f}(a_\varepsilon, b_\varepsilon) \ge C^{-2} i_{p,f}(a_\varepsilon, b_\varepsilon) \\
&= C^{-2}[d(a_\varepsilon, b_\varepsilon) \wedge d(a_\varepsilon, p)]f(d(a_\varepsilon, p)).
\end{aligned}
$$

Letting $\varepsilon \to 0$, and using continuity of $f$, we deduce that

$$
\begin{aligned}
\operatorname{diam}_{p,f} \hat{X}_p &\ge C^{-2} \left[ \frac{\operatorname{diam}(X_p)}{2} \wedge \delta \right] f(\delta) \\
&= C^{-2} \left( \left[ \frac{\operatorname{diam}(X_p) f(\delta)}{2} \right] \wedge F(\delta) \right). \qquad \square
\end{aligned}
$$

There are also generalized versions of the later results in Section 3. We leave this task mainly to the reader, with a few exceptions. First we establish a generalized version of Proposition 3.3 because there is a wrinkle in the result compared with the original version: the composition of a pair of generalized inversions gives a bilipschitz mapping as long as the associated inversion functions are in some sense dual to each other.

For duality, we only consider unbounded spaces. We say that two admissable inversion functions $f_i : (0,\infty) \to (0,\infty)$, $i = 1, 2$, are $C$-quasidual, $C \geq 1$, if the associated functions $F_i(t) := tf_i(t)$ satisfy the conditions

$$\frac{t}{C} \leq F_1(F_2(t)) \leq Ct \quad \text{and} \quad \frac{t}{C} \leq F_2(F_1(t)) \leq Ct.$$

Note that when $t \simeq s$, $f_1(t)f_2(F_1(s)) \simeq 1$. A pair of 1-quasidual admissable inversion functions is given by $f_1(t) = t^{-\alpha}$ and $f_2(t) = t^{-\alpha/(\alpha-1)}$ for $\alpha > 1$.

In the following proof, an inequality of the form $A \lesssim B$ between two non-negative quantities $A$, $B$ means that $A \leq C_0 B$, where $C_0$ depends only on the constants $C$, $C'$ in the statement of the lemma, and $A \simeq B$ means that $A \lesssim B \lesssim A$. Recall that for an unbounded space $X$ we let $p' \in \operatorname{Inv}_{p,f}(X)$ correspond to the point at infinity.

***Lemma 7.2.*** *Let $(X,d)$ be an unbounded metric space. Suppose $f_1$, $f_2$ are $C$-admissable inversion functions that are $C'$-quasidual, where $C > 2$, $C' \geq 1$. Fix $p \in X$ and let $d' = (d_{p,f_1})_{p',f_2}$ denote the distance on $X' = \operatorname{Inv}_{p',f_2}(\operatorname{Inv}_{p,f_1} X)$. There exists a constant $C'' = C''(C,C') > 0$ such that:*

(a) *if $p$ is a non-isolated point, then the identity map $(X,d) \xrightarrow{\text{id}} (X,d') = (X',d')$ (where $p \mapsto p''$) is $C''$-bilipschitz;*

(b) *if $p$ is an isolated point, then the identity map $(X_p,d) \xrightarrow{\text{id}} (X_p,d')$ is $C''$-bilipschitz.*

*Proof.* We associate $F_1$, $F_2$ with $f_1$, $f_2$, respectively. For brevity, we write $|z| = d(z,p)$. Suppose $x, y \in X_p$, with $|x| \leq |y|$. Then

$$d_{p,f_1}(x,y) = [d(x,y) \wedge |x|]f_1(|x|).$$

By (I-1) and (I-0), $F_1(|y|) \lesssim F_1(|x|)$ and $f_2(F_1(|x|) \wedge F_1(|y|)) \simeq f_2(F_1(|y|))$, so by (7.1)

$$\begin{aligned} d'(x,y) &\simeq [i_{p,f_1}(x,y) \wedge F_1(|y|)] \cdot f_2(F_1(|y|)) \\ &\simeq [[(d(x,y) \wedge |x|)f_1(|x|)] \wedge F_1(|y|)] \cdot f_2(F_1(|y|)). \end{aligned} \tag{7.2}$$

Suppose $d(x,y) \leq |x|/2$, so $|y| \simeq |x|$ and $f_1(|x|) \simeq f_1(|y|)$. Then

$$\begin{aligned} d'(x,y) &\simeq [[d(x,y)f_1(|x|)] \wedge F_1(|y|)] \cdot f_2(F_1(|y|)) \\ &\simeq [d(x,y)f_1(|x|)f_2(F_1(|y|))] \wedge |y|, \end{aligned}$$

where the last inequality follows by distributivity and quasiduality. Now quasiduality and $|x| \simeq |y|$ imply that $f_1(|x|)f_2(F_1(|y|)) \simeq 1$, so $d'(x,y) \simeq d(x,y) \wedge |y| = d(x,y)$ as required.

Suppose $d(x,y) \geq |x|/2$. Then $d(x,y) \simeq |y|$ (this is clear if $d(x,y) \geq 2|x|$; for intermediate values of $d(x,y)$, use the assumption $|y| \geq |x|$). As before, $F_1(|y|) \lesssim F_1(|x|)$, so it follows from (7.2) and quasiduality that

$$d'(x,y) \simeq (F_1(|x|) \wedge F_1(|y|)) \cdot f_2(F_1(|y|))$$
$$\simeq F_1(|y|) f_2(F_1(|y|)) \simeq |y| \simeq d(x,y).$$

Finally, to prove that $d'(x,p'') \simeq d(x,p)$ when $p$ is non-isolated, we observe that

$$d'(x,p'') \simeq F_2(i_{p,f_1}(x,p')) = F_2(F_1(|x|)) \simeq |x|. \qquad \square$$

In the following, $\delta_{p,f}(\cdot)$ is the $d_{p,f}$-distance to the boundary $\partial_{p,f}\Omega$ of $\Omega$ as a subspace in $\mathrm{Inv}_{p,f}(X)$. Its proof is similar to that of Lemma 3.6 and so is left to the reader.

***Lemma 7.3.*** *Let $\Omega \subset X_p = X \setminus \{p\}$ (for some fixed base point $p \in X$) be an open subspace of $(X,d)$, and let $f$ be a $C$-admissable inversion function, $C > 2$. Then for all $x \in \Omega$:*

(a) $\delta_{p,f}(x) \geq (F(d(x,p)) \wedge [\delta(x) f(d(x,p))])/C^3$,
(b) $\delta(x) \geq (d(x,p) \wedge [\delta_{p,f}(x)/f(d(x,p))])/C$.

We state but do not prove a generalized version of Proposition 4.2.

***Proposition 7.4.*** *If $(X,d)$ is locally $c$-quasiconvex, $p \in X$, and $f$ is a $C$-admissable inversion function, then $(X_p, d_{p,f})$ is locally $b$-quasiconvex for all $b > Cc$.*

Finally, here is a generalized version of Theorem 4.6. Below $k_{p,f}$ denotes quasihyperbolic distance for $\Omega$ as a subspace in $\mathrm{Inv}_{p,f}(X)$.

***Theorem 7.5.*** *Let $(X,d)$ be complete and fix a base point $p \in X$. Let $f$ be a $C$-admissable inversion function. Suppose $\Omega \subset X_p$ is a locally compact, open, locally $c$-quasiconvex subspace with $\partial\Omega \neq \emptyset \neq \partial_{p,f}\Omega$. Then the identity map $\mathrm{id} : (\Omega,k) \to (\Omega, k_{p,f})$ is $M$-bilipschitz, where $M = cC[a \vee (2bC^2)]$, $b$ is as in Theorem 4.6,*

$$a = \begin{cases} 1 & \text{if } \Omega \text{ is unbounded,} \\ CD^{\log_2(C/2)} & \text{if } \Omega \text{ is bounded,} \end{cases}$$

*and $D = 1 + \mathrm{diam}\,\Omega/[\mathrm{d}(p,\partial\Omega) \vee (\mathrm{diam}(\partial\Omega)/2)]$.*

*Proof.* The proof is broadly similar to that of Theorem 4.6. Arguing as there, we see that

$$L(x,\mathrm{id}) \le L(x,j)\cdot L(x,h)\cdot L(x,i^{-1}) \le \frac{2cC}{\delta_{p,f}(x)}\cdot f(d(x,p))\cdot \delta(x),$$

$$L(x,\mathrm{id}^{-1}) \le L(x,i)\cdot L(x,h^{-1})\cdot L(x,j^{-1}) \le \frac{c}{\delta(x)}\cdot \frac{1}{f(d(x,p))}\cdot \delta_{p,f}(x).$$

Thus, it suffices to show that $\delta(x)f(d(x,p)) \simeq \delta_{p,f}(x)$; more precisely, we must establish

$$\forall\, x\in\Omega: \quad \begin{aligned} &2cC\delta(x)f(d(x,p)) \le M\delta_{p,f}(x)\,,\\ &c\delta_{p,f}(x) \le M\delta(x)f(d(x,p)). \end{aligned}$$

Recalling the definition of $M$ and the estimates from Lemma 7.3 we see that the above inequalities are equivalent to

$$\forall\, x\in\Omega: \quad \begin{aligned} &\delta(x) \le bd(x,p)\,,\\ &\delta_{p,f}(x) \le aF(d(x,p)). \end{aligned}$$

The first estimate was proved in Theorem 4.6, so it suffices to show $\delta_{p,f}(x) \le aF(d(x,p))$.

If $\Omega$ is unbounded, then $p'\in\partial_{p,f}(\Omega)$ and $\delta_{p,f}(x)\le i_{p,f}(x,p') = F(d(x,p))$. We assume $\Omega$ is bounded. (In this case, $\partial\Omega\ne\{p\}$ because $\partial_{p,f}\Omega\ne\varnothing$.) Pick $q\in\partial\Omega$ so that $d(p,q)\ge \mathrm{d}(p,\partial\Omega)\vee\frac{1}{2}\operatorname{diam}(\partial\Omega)$. Then

$$\delta_{p,f}(x)\le i_{p,f}(x,q)\le F(d(x,p)\wedge d(q,p)).$$

If $d(x,p)\le d(q,p)$, then $\delta_{p,f}(x)\le F(d(x,p))$. If $d(q,p)<d(x,p)\le 2d(q,p)$, then

$$\delta_{p,f}(x)\le F(d(q,p))\le CF(d(x,p)).$$

Finally, suppose $d(x,p)\ge 2d(q,p)$ and let $n\in\mathbb{N}$ be such that

$$2^n\le \frac{d(x,p)}{d(q,p)} < 2^{n+1}.$$

Appealing to (I-0) $n+1$ times we obtain

$$f(d(q,p))\le Cf(2d(q,p))\le\cdots\le C^n f(2^n d(q,p))\le C^{n+1}f(d(x,p))$$

which yields $\delta_{p,f}(x)\le F(d(q,p))\le C(C/2)^nF(d(x,p))\le aF(d(x,p))$, where the last inequality holds because

$$2^n\le\frac{d(x,p)}{d(q,p)}\le 1+\frac{d(x,q)}{d(q,p)}\le 1+\frac{\operatorname{diam}\Omega}{\mathrm{d}(p,\partial\Omega)\vee 1/2\operatorname{diam}(\partial\Omega)}. \qquad \square$$

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