Inversion and Sphericalization Studied Through the Ferrand Metric

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Abstract

In this paper we extend upon the work done in Buckley et al. (2008) and Herron and Poranee (2013) where metric space inversions and sphericalizations are investigated using the quasihyperbolic metric. Here, we present an analogous study of metric space inversions and sphericalizations, but this time using the Ferrand metric. Using what we find for the Ferrand metric we also present new estimates for the stretching of the quasihyperbolic metric when it is inverted. We also present some methods of calculating or estimating the Ferrand metric using properties of the metric space itself.

1 Introduction and Definitions

In this section I will go through a brief introduction to inversions, sphericalizations, the Ferrand metric, and Appolonian objects.

An inversion of a metric space (X, d) through the point o is the metric space $(X_o = X\backslash\{o\}, d_o)$ where d_o is defined through the following equations.

$$i_o(x, y) \equiv \frac{d(x, y)}{d(x, o)d(y, o)} \tag{1}$$

$$d_o(x, y) \equiv inf\{\Sigma_{i=1}^{k} i_o(x_i, x_{i-1} : x = x_0, ..., x_k = y \in X_o\} \tag{2}$$

where the infimum is taken over all chains $x_o, ..., x_k$. This new distance function can be pictured as taking the space X and inverting it through the point o so that the distance between two points that are far away from o is small, but the distance between two points which are close to o is large (and increasingly large the closer you get to o). From these definitions, one can show that $\forall x, y \in X$

$$\frac{1}{4}i_o(x,y) \leq d_o(x,y) \leq i_o(x,y) \leq \frac{1}{d(x,o)} + \frac{1}{d(y,o)} \tag{3}$$

A sphericalization of a metric space (X, d) through the point o is defined in an analogous way. The sphericalization of (X, d) is $(X_o, \hat{d})$ where $\hat{d}$ is defined below.

$$s_o(x,y) \equiv \frac{d(x,y)}{(1+d(x,o))(1+d(y,o))} \tag{4}$$

$$\hat{d}(x,y) \equiv inf\{\Sigma_{i=1}^{k} s_o(x_i, x_{i-1} : x = x_0, ..., x_k = y \in X_o\} \tag{5}$$

From these definitions, one can show that $\forall x, y \in X$

$$\frac{1}{4}s_o(x,y) \leq \hat{d}(x,y) \leq s_o(x,y) \leq \frac{1}{1+d(x,o)} + \frac{1}{1+d(y,o)} \tag{6}$$

Using inversions and sphericalizations, we can make estimates of the Ferrand distance between two points, which is defined below.

$$\Phi(x,y) \equiv \inf_{\gamma: x \curvearrowright y} \int_\gamma \varphi(x) ds \tag{7}$$

$$\varphi(x) \equiv \sup_{a,b \in \partial\Omega} \frac{d(a,b)}{d(a,x)d(b,x)} \tag{8}$$

where in the above definition, $\Omega \subset X$ such that Ω has at least two boundary points (because without two boundary points, we cannot defined φ in the way we have). Another important metric for studying the Ferrand metric is the quasi-hyperbolic metric, $k(x, y)$. This metric is defined below for a set $\Omega \subset X$.

$$k(x,y) \equiv \inf_{\gamma: x \curvearrowright y} \int_\gamma \delta(x) ds \tag{9}$$

$$\delta(x) \equiv d(x, \partial\Omega) = \inf_{a \in \partial\Omega} d(x,a) \tag{10}$$

An important result of comparing δ to φ is that $\forall x \in \Omega$

$$\frac{c}{\delta(x)} \leq \varphi(x) \leq \frac{2}{\delta(x)} \tag{11}$$

where $c \equiv \frac{1}{2}\frac{diam(\Omega)}{diam(\partial\Omega)}$.

Above we defined $\varphi(x)$ using a supremum over points in the boundary of Ω, but as we will see later, we can actually also define φ using a supremum over points in the complement of Ω. I will not address this until later, however. In fact, in order to show this, we must first define what are called Apollonian balls and Apollonian spheres, which I will do now. The Apollonian sphere and Apollonian ball with limit points p, q through the point ζ (where p, q, ζ are all distinct points) is defined below.

$$AS(p, q|\zeta) \equiv \{x \in X : |x, p, q, \zeta| = 1\} \tag{12}$$

$$AB(p, q|\zeta) \equiv \{x \in X : |x, p, q, \zeta| < 1\} \tag{13}$$

Equivalently, we can define it in the more useful form (for our purposes) shown below.

$$AS(p, q|\zeta) \equiv \{x \in X : d(x, p) = td(x, q)\} \tag{14}$$

$$AB(p, q|\zeta) \equiv \{x \in X : d(x, p) < td(x, q)\} \tag{15}$$

where the parameter t is found by plugging ζ in for x. If we want to talk about a collection of Apollonian spheres or balls determined by two points (we need three to determine a unique AS), then we denote this by AS(p,q;t) or AB(p,q;t), respectively. t here is determined by a third point in the way described above. We have now defined all the key quantities for this study and we can now move on to proving some things using these definitions.

2 Inversions

Theorem 2.1. *Let (X, d) be a geodesic metric space and let $\Omega \subset X_o = X \backslash \{o\}$ (where o is thought of as a generic base point which we invert through) be an open subspace of X with at least two boundary points. Then the following inequality holds for all $x \in \Omega$:*

$$\frac{|x|^2}{4}\varphi(x) \leq \varphi_o(x) \leq 16|x|^2\varphi(x)$$

where $|x| \equiv d(x, o)$.

Proof. Let $a, b \in \Omega$ be phi-extremal points. That is, let $\varphi(x) = \tau^{ab}(x)$. Then we automatically get the following

$$\varphi_o(x) = \sup_{\xi,\eta\in\partial\Omega} \tau_o^{\xi\eta}(x) \geq \tau_o^{ab}(x) = \frac{d_o(a,b)}{d_o(a,x)d_o(b,x)} \tag{16}$$

Using (3), the above implies

$$\varphi_o(x) \geq \frac{i_o(a,b)}{4i_o(a,x)i_o(b,x)} = \frac{d(a,b)}{4d(a,x)d(b,x)}|x|^2 = \frac{1}{4}\varphi(x)|x|^2 \tag{17}$$

This establishes the first inequality. Now let's flip the argument around and get the second inequality. Let $a, b \in \Omega$ now be φ_o-extremal points. That is, let $\varphi_o(x) = \tau_o^{ab}(x)$. Now we get

$$\varphi(x) \geq \frac{d(a,b)}{d(a,x)d(b,x)} = \frac{d(a,b)}{d(a,x)d(b,x)} \frac{d(o,a)d(o,b)d(o,x)^2}{d(o,a)d(o,b)d(o,x)^2} \tag{18}$$

Now regroup the term on the right hand side and use (3) again to obtain

$$\varphi(x) \geq \frac{i_o(a,b)}{i_o(a,x)i_o(b,x)} d(x,o)^{-2} \geq \frac{1}{16}\varphi_o(x)|x|^{-2} \tag{19}$$

□

Lemma 2.2. *ds_o is related to ds in the following way*

$$ds_o = \frac{ds}{|x|^2}$$

Proof. To see this, we need to consider how paths are stretched/squished when we invert our metric space. Consider the maximal and minimal stretching estimates given below:

$$L(x,f) \equiv \limsup_{y\to x} \frac{d(f(x),f(y))}{d(x,y)} \tag{20}$$

$$l(x,f) \equiv \liminf_{y\to x} \frac{d(f(x),f(y))}{d(x,y)} \tag{21}$$

Buckley et al. (2008) gives us that the identity map $(X_o, d) \to (X_o, d_o)$ satisfies $L(x, id) = l(x, id) = \frac{1}{d(x,o)^2}$ for every non-isolated point $x \in X$. This proves the Lemma.

□

We will now use Lemma 2.2 and Theorem 2.1 together to show Theorem 2.3.

Theorem 2.3. *Let (X, d) be a complete metric space and fix a base point $o \in X$. Also, let Ω be an open subspace of X_o. Then the identity map $(\Omega, \Phi) \rightarrow^{id} (\Omega, \Phi_o)$ is 16-bilipschitz.*

Proof. Theorem 2.1 says that

$$16\varphi(x)|x|^2 \geq \varphi_o(x) \geq \frac{1}{4}\varphi(x)|x|^2 \tag{22}$$

$$16\varphi(x)|x|^2 ds_o \geq \varphi_o(x) ds_o \geq \frac{1}{4}\varphi(x)|x|^2 ds_o \tag{23}$$

We now make use of Lemma 2.2 and see that the above implies

$$16\varphi(x)|x|^2 \frac{ds}{|x|^2} \geq \varphi_o(x) ds_o \geq \frac{1}{4}\varphi(x)|x|^2 \frac{ds}{|x|^2} \geq \frac{1}{4}\varphi(x)|x|^2 \frac{ds}{4|x|^2} \tag{24}$$

$$16\varphi(x) ds \geq \varphi_o(x) ds_o \geq \frac{1}{16}\varphi(x) ds \tag{25}$$

We now recognize that since $\Phi(x,y) \doteq \inf_{\gamma:x\frown y} \int_\gamma \varphi(x) ds$ and $\Phi_o(x,y) \doteq \inf_{\gamma:x\frown y} \int_\gamma \varphi_o(x) ds_o$, that the above inequality immediately implies that

$$16\Phi(x,y) \geq \Phi_o(x,y) \geq \frac{1}{16}\Phi(x,y) \forall x, y \in \Omega. \tag{26}$$

□

We are now in a position to use Theorem 2.3 to obtain a new estimate on the stretching of the quasihyperbolic metric when it goes through an inversion. This statement is made precise in the following theorem.

Theorem 2.4. *Let (X, d) be a complete metric space and fix a base point $o \in X$. Let Ω be an open subspace of X_o. Then the identity map $(\Omega, k) \rightarrow^{id} (\Omega, k_o)$ is $\frac{32}{c}$-bilipschitz where $c \equiv \frac{1}{2}\left(\frac{diam(\Omega)}{diam(\partial\Omega)}\right)$*

Proof. We know the following:

$$\frac{c}{\delta} \leq \varphi \leq \frac{2}{\delta} \tag{27}$$

and

$$\frac{c}{\delta_o} \leq \varphi_o \leq \frac{2}{\delta_o} \tag{28}$$

(27) implies that

$$\frac{c}{\varphi} \leq \delta \leq \frac{2}{\varphi} \tag{29}$$

$$\frac{\varphi}{2}ds \leq \frac{ds}{\delta} \leq \frac{\varphi}{c}ds \tag{30}$$

We can now make use of Theorem 2.3 to get

$$\frac{\varphi_o ds_o}{32} \leq \frac{\varphi ds}{2} \leq \frac{ds}{\delta} \leq \frac{\varphi ds}{c} \leq \frac{16\varphi_o ds_o}{c} \tag{31}$$

Now use (28) to get

$$\frac{cds_o}{32\delta_o} \leq \frac{ds}{\delta} \leq \frac{32ds_o}{c\delta_o} \tag{32}$$

□

It immediately follows that the identity map $(\Omega, k) \to^{id} (\Omega, k_o)$ is $\frac{32}{c}$-bilipschitz.

3 Sphericalizations

Theorem 3.1. *Let (X, d) be a geodesic metric space and let $\Omega \subset X_o = X \backslash \{o\}$ (where o is thought of as a generic base point which we invert through) be an open subspace of X with at least two boundary points. Then the following inequalities hold for all $x \in \Omega$:*

$$\varphi(x)(\frac{1+|x|^2}{4}) \leq \hat{\varphi}(x) \leq 16(1+|x|^2)\varphi(x)$$

where $|x| \equiv d(x, o)$.

Proof. Let $a, b \in \Omega$ be phi-extremal points. That is, let $\varphi(x) = \tau^{ab}(x)$. Then we automatically get the following

$$\hat{\varphi}(x) \geq \frac{\hat{d}(a,b)}{\hat{d}(a,x)\hat{d}(b,x)}. \tag{33}$$

We have the following fact from Buckley et al. (2008) which we will make use of.

$$\frac{1}{4}s_o(x,y) \le \hat{d}(x,y) \le s_o(x,y) \tag{34}$$

where s_o is defined as follows.

$$s_o(x,y) = \frac{d(x,y)}{(1+d(x,p))(1+d(y,p))} \tag{35}$$

Now, using (33), (34), and (35) we can get that

$$\hat{\varphi}(x) \ge \frac{\frac{1}{4}s_o(a,b)}{s_o(a,x)s_o(b,x)} \tag{36}$$

$$\implies \hat{\varphi}(x) \ge \frac{1}{4}\left(\frac{d(a,b))}{(1+d(a,o))(1+d(b,o))}\frac{(1+d(a,o))(1+d(x,o))(1+d(b,o))(1+d(x,o))}{d(a,x)d(b,x)}\right) \tag{37}$$

$$\implies \hat{\varphi}(x) \ge \frac{1}{4}\frac{d(a,b)}{d(a,x)d(b,x)}(1+|x|)^2 \tag{38}$$

$$\implies \hat{\varphi}(x) \ge \frac{1}{4}\varphi(x)(1+|x|)^2 \tag{39}$$

So this establishes the first inequality in our theorem. For the other inequality, just flip this argument around. Assume that we have points a and $b \in \Omega$ such that $\hat{\varphi}(x) = \hat{\tau}^{ab}(x)$. Then we get automatically that

$$\varphi(x) \ge \frac{d(a,b)}{d(a,x)d(b,x)} \tag{40}$$

Now we can multiply on top and bottom by a number of different factors such that we can pull out factors of the s_o distance between two points.

$$\varphi(x) \ge \frac{d(a,b)(1+d(a,o))(1+d(b,o))(1+d(a,o))(1+d(x,o))(1+d(b,o))(1+d(x,o))}{d(a,x)d(b,x)(1+d(a,o))(1+d(b,o))(1+d(a,o))(1+d(x,o))(1+d(b,o))(1+d(x,o))} \tag{41}$$

$$\implies \varphi(x) \ge \frac{s_o(a,b)}{s_o(a,x)s_o(b,x)}\frac{1}{(1+d(x,o))^2} \tag{42}$$

$$\implies \varphi(x) \ge \frac{\hat{d}_o(a,b)}{16\hat{d}_o(a,x)\hat{d}_o(b,x)}\frac{1}{(1+|x|)^2} \tag{43}$$

$$\implies \varphi(x) \ge \frac{\hat{\varphi}(x)}{16(1+|x|)^2} \tag{44}$$

□

We will now look at how $\hat{ds}$ relates to ds.

Lemma 3.2. *$\hat{ds}$ is related to ds in the following way:*

$$\hat{ds} = \frac{ds}{(1+|x|)^2}$$

Proof. This proof is left to the reader, but can be shown in an analogous way to how Buckley et al. (2008) show Lemma 2.2. □

We now want to use Lemma 3.2 and Theorem 3.1 together to prove Theorem 3.3.

Theorem 3.3. *Let (X, d) be a complete metric space and fix a base point $o \in X$. Also, let Ω be an open subspace of X_o. Then the identity map $(\Omega, \Phi) \to^{id} (\Omega, \hat{\Phi})$ is 16-bilipschitz.*

Proof. Theorem 3.1 says that

$$\frac{\varphi(x)(1+|x|)^2}{4} \leq \hat{\varphi}(x) \leq 16(1+|x|)^2\varphi(x) \tag{45}$$

$$\implies \frac{\varphi(x)(1+|x|)^2}{4}\hat{ds} \leq \hat{\varphi}(x)\hat{ds} \leq 16(1+|x|)^2\varphi(x)\hat{ds} \tag{46}$$

Now we can use Lemma 3.2 to get the following.

$$\frac{\varphi(x)(1+|x|)^2}{4}\frac{ds}{4(1+|x|)^2} \leq \frac{\varphi(x)(1+|x|)^2}{4}\frac{ds}{(1+|x|)^2} \leq \hat{\varphi}\hat{ds} \leq 16(1+|x|)^2\varphi(x)\frac{ds}{(1+|x|)^2} \tag{47}$$

$$\implies \frac{\varphi ds}{16} \leq \hat{\varphi}\hat{ds} \leq 16\varphi ds \tag{48}$$

We now recognize that since $\Phi(x,y) \doteq \inf_{\gamma: x \frown y} \int_\gamma \varphi(x)ds$ and $\hat{\Phi}(x,y) \doteq \inf_{\gamma: x \frown y} \int_\gamma \hat{\varphi}(x)\hat{ds}$, that the above inequality immediately implies that

$$16\Phi(x,y) \geq \hat{\Phi}(x,y) \geq \frac{1}{16}\Phi(x,y) \forall x, y \in \Omega. \tag{49}$$

□

Now we can use Theorem 3.3 to get a new estimate on the stretching of the quasihyperbolic metric when it goes through a sphericalization. This is spelled out in Theorem 3.4

Theorem 3.4. *Let (X,d) be a complete metric space and fix a base point $o \in X$. Let Ω be an open subspace of X_o. Then the identity map $(\Omega, k) \rightarrow^{id} (\Omega, \hat{k})$ is $\frac{32}{c}$-bilipschitz where $c \equiv \frac{1}{2}(\frac{diam(\Omega)}{diam(\partial\Omega)})$*

Proof. We know the following:

$$\frac{c}{\delta} \leq \varphi \leq \frac{2}{\delta} \tag{50}$$

$$\frac{c}{\hat{\delta}} \leq \hat{\varphi} \leq \frac{2}{\hat{\delta}} \tag{51}$$

(50) gives us that

$$\frac{c}{\varphi} \leq \delta \leq \frac{2}{\varphi} \tag{52}$$

$$\implies \frac{\varphi}{2} ds \leq \frac{ds}{\delta} \leq \frac{\varphi}{c} ds \tag{53}$$

We can now make use of Theorem 3.3 to get

$$\frac{\hat{\varphi}\hat{ds}}{32} \leq \frac{\varphi ds}{2} \leq \frac{ds}{\delta} \leq \frac{\varphi ds}{c} \leq \frac{16\hat{\varphi}\hat{ds}}{c} \tag{54}$$

Now, (51) can be used to get

$$\frac{c\hat{ds}}{32\hat{\delta}} \leq \frac{ds}{\delta} \leq \frac{32\hat{ds}}{c\hat{\delta}} \tag{55}$$

□

It immediately follows that the identity map $(\Omega, k) \rightarrow^{id} (\Omega, \hat{k})$ is $\frac{32}{c}$-bilipschitz.

4 Estimates for $\varphi(x)$

Theorem 4.1. *Suppose (X,d) is a metric space and $\Omega \subset X_o$ is an open subspace of X_o. Then*

$$d_o diameter\ of\ \partial\Omega \leq \varphi(o) \leq 4(d_o diameter\ of\ \partial\Omega).$$

Proof. Notice that (3) implies that for any $a, b \in X_o$,

$$\frac{1}{4}\frac{d(a,b)}{d(a,o)d(b,o)} \leq d_o(a,b) \leq \frac{d(a,b)}{d(a,o)d(b,o)} \tag{56}$$

But the quantities on the right and left of this inequality are just $\tau^{ab}(o)$. Thus,

$$\frac{1}{4}\tau^{ab}(o) \leq d_o(a,b) \leq \tau^{ab}(o) \tag{57}$$

$$\frac{1}{4}\sup_{a,b\in\partial\Omega} \tau^{ab}(o) \leq \sup_{a,b\in\partial\Omega} d_o(a,b) \leq \sup_{a,b\in\partial\Omega} \tau^{ab}(o) \tag{58}$$

$$\frac{1}{4}\varphi(o) \leq d_o\text{diameter of } \partial\Omega \leq \varphi(o) \tag{59}$$

$$\frac{1}{4d_o\text{diameter of } \partial\Omega} \leq \frac{1}{\varphi_o} \leq \frac{1}{d_o\text{diameter of } \partial\Omega} \tag{60}$$

$$d_o\text{diameter of } \partial\Omega \leq \varphi(o) \leq 4(d_o\text{diameter of } \partial\Omega) \tag{61}$$

□

We will now establish that, even if we define $\varphi(x)$ using points in the complement of our set, that the supremum of $\tau^{ab}(x)$ is established on the boundary of the set. Therefore, defining $\varphi(x)$ using points in the complement and defining it using points in the boundary are equivalent ways of defining φ.

Theorem 4.2. *The definitions* $\varphi(x) \equiv \sup_{a,b\in\overline{\Omega}} \tau^{ab}(x)$ *and* $\varphi(x) \equiv \sup_{a,b\in\partial\Omega} \tau^{ab}(x)$ *are equivalent, where* $\overline{\Omega} = X\backslash\Omega$ *is the complement of* Ω.

Proof. We will show this by showing that $\tau^{ab}(x)$ is maximized strictly when a and b are in the boundary. As usual, begin with a geodesic metric space (X, d) and $\Omega \subset X$ where the set $\overline{\Omega} = X\backslash\Omega$ contains at least two points. Consider $a, b \in \overline{\Omega}$, $x \in \Omega$, and the collection of Apollonian spheres $AS(x, b; t)$. It can be easily shown that $t < t' \Leftrightarrow AS(x, b; t) \subset AB(x, b; t')$. This is an important fact to keep in mind during this proof.

Now think about a path, $\gamma : [0, 1] \to X$, where $\gamma(0) = x$ and $\gamma(1) = a$. There exists a unique $\zeta \in |\gamma|$ such that $AS(x, b|\zeta) \cap \partial\Omega \neq \emptyset$ and $AS(x, b|\zeta) \subset \Omega$. This should make intuitive sense if you think about expanding spheres. Begin with an Apollonian sphere with very small parameter t so that the entire sphere is contained within Ω. Now start increasing t (this means expanding the sphere) until the sphere finally touches the boundary of Ω.

Then, since γ is a path which begins inside Ω and ends outside Ω, there must be a first point, ζ on that path where the sphere through x, b, ζ intersects the boundary of Ω and where the Apollonian ball through x, b, ζ is still entirely contained within Ω.

Now notice that we can write the Apollonian parameter, t, in terms of $\tau^{\zeta b}(x)$. By definition, for an Apollonian sphere $AS(x, b|\zeta)$, we have our parameter $t = \frac{d(x,\zeta)}{d(b,\zeta)}$. Therefore, $t = \frac{1}{\tau^{\zeta b}(x)d(x,b)}$. Now, we know that a is not contained within $AB(x, b|\zeta)$ since this ball is entirely within Ω, and a, as we have constructed it, is in the complement of Ω. Therefore, if t' is the Apollonian parameter such that $AS(x, b; t') = AS(x, b|a)$, then we have $AS(x, b|\zeta) = AS(x, b; t) \subset AB(x, b; t')$. But, as mentioned above, this immediately implies that $t < t'$. Now we just write t and t' in terms of $\tau^{\zeta b}(x)$ and $\tau^{ab}(x)$, respectively, and we get the following.

$$t = \frac{1}{\tau^{\zeta b}(x)d(x,b)} \tag{62}$$

$$t' = \frac{1}{\tau^{ab}(x)d(x,b)} \tag{63}$$

We've got that $t < t'$, therefore, $\frac{1}{\tau^{\zeta b}(x)d(x,b)} < \frac{1}{\tau^{ab}(x)d(x,b)}$. Cancelling factors of $d(x, b)$ and inverting the inequality, we get that $\tau^{ab}(x) < \tau^{\zeta b}(x)$.

Now we can go through an analogous process looking at Apollonian objects with limit points x and ζ and we can show that we can increase τ by using a point on the boundary of Ω rather than the point b. This proves the theorem.

□

References

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