Exploring the SU(N) Super Yang-Mills Moduli Space of Vacua Through Isogenies Between Abelian Varieties

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***Abstract*—The moduli space of vacua is the term used to describe the space of all the possible ground states of an N=4 supersymmetric Yang-Mills theory. In this paper, I will describe this space of vacua and explore the properties of the theories which this space describes. In particular, we will attempt to understand what happens when we take limits in this space in a specific way, corresponding to taking isogenies between the abelian varieties associated to each point in the space.**

I. Introduction

Throughout this paper, the term group will be used heavily as well as some elementary group theory. It is not the goal of this paper to describe group theory, and therefore, I will simply refer the reader to [1] to learn more about the definition of groups and some of their properties. Once a familiarity with groups is established, all the reader will need to know is physics at the undergraduate level in order to understand this paper. To begin with, I will describe the physics setting that we are working in. Through this physics setting, I will show the importance of complex tori when talking about the moduli space of vacua in an N=4 super Yang-Mills theory. Once this importance is established, I will go on to talk about polarizations of lattices and their connections to abelian varieties before finally describing isogenies between abelian varieties. Isogenies between abelian varieties and the induced polarization due to an isogeny is the topic of my current research with Dr. Argyres which will be continued at least through the Spring 2021 semester.

A. Yang-Mills Theory

Yang-Mills theories are basic generalizations of electromagnetism which arise from an underlying Lie group. Lie groups are nothing more than a continuous group with a smooth manifold structure. Each Lie group has an associated Lie algebra, which can be thought of as the tangent space to the Lie group at the identity element of the Lie group. This is shown symbolically in eqn. 1 where G is the Lie group and g is the Lie algebra.

$$g = T_{\mathbb{I}}(G) \tag{1}$$

Every compact Lie group then has an associated Yang-Mills theory which is constructed by taking a basis of the Lie algebra and associating each basis vector with an operator whose role is to create a massless spin-1 particle. The states of these particles live in a Hilbert space which is infinite dimensional. The basis of this Hilbert space consists of states that represent any number of massless spin-1 particles in a given system. So, inside of this Hilbert space there lives a basis vector representing the state of a single massless spin-1 particle. There is also a basis vector representing the state of two massless spin-1 particles, and three, and four, and so on to infinity. To understand what this means physically, recall that massless spin-1 particles in quantum electromagnetism are photons. So in the case of a one-dimensional Lie algebra, we have one operator which creates photons and the states of these photons live in a Hilbert space whose basis vectors describe an arbitrary number of photons living in a region of space.

This is just the simple case of a one-dimensional Lie algebra, however. In the more general case of an N-dimensional Lie algebra, we get N different operators that each create their own spin-1 particle (analogue of a photon). In other words, if we have an N-dimensional Lie algebra, then our Yang-Mills theory associated to this Lie algebra has N different kinds of photons which all couple to matter with specific strengths! To make this even more general, consider a Lie group G such that $dim(G) = D$ and $rank(G) = N$. The rank of a group G is the smallest cardinality for a generating set of G. That is, it is the smallest number of independent members of the group such that combinations of these independent members (and their inverses) can generate all the rest of the members of the group. Now, the Yang-Mills theory associated to the gauge group G has D massless spin-1 particles, of which N are neutral (i.e. they don't interact with each other and so are analogous to photons) and the remaining $D - N$ have charges with respect to each of the N "photons".

To give us a better idea of what was said in the preceding paragraphs, let's return to the case of a one-dimensional Lie algebra and in particular, we'll pick the so-called $u(1)$ Lie algebra. The $U(1)$ Lie group associated with the $u(1)$ Lie

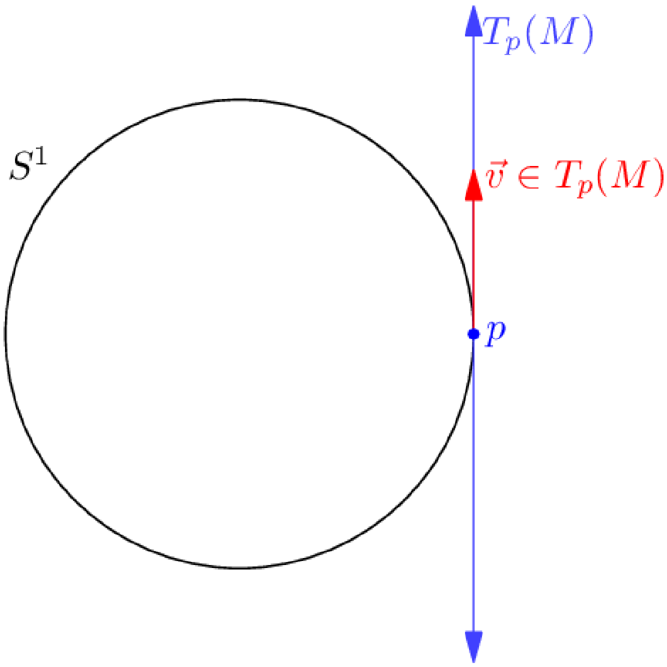


Fig. 1: This shows visually the $U(1)$ Lie group and its associated Lie algebra $u(1)$. $U(1)$ is isomorphic to the circle S^1 and $u(1)$ is isomorphic to the real line $\mathbb{R}$. This figure is taken from [2].

algebra consists of all 1 by 1 unitary matrices. That is, it consists of all complex numbers such that the number times its complex conjugate is equal to one. Thus, the elements of the $U(1)$ Lie group are simply phases; that is, any element $\lambda \in U(1)$ can be written as $\lambda = e^{i\phi}$ for some $\phi \in \mathbb{R}$. So we have learned that $U(1)$ is isomorphic to the circle S^1. Thinking of $u(1)$ then as the tangent space to $U(1)$ at the identity element of $U(1)$ (that is, at the point $(1, 0) \in \mathbb{R}^2$), we see that $u(1)$ is just the real line $\mathbb{R}$. This is shown visually in fig. 1. Thus the Lie algebra $u(1)$ has just one basis element and the Yang-Mills theory associated to this Lie algebra has just one operator that creates massless spin-1 particles. This operator turns out to transform as a lorentz 4-vector and in the classical limit, is the familiar vector potential from electromagnetism. This is shown in eqn. 2.

$$\hat{A}^\mu(\vec{x}) \to (\Phi(\vec{x}), \vec{A}(\vec{x})) \tag{2}$$

The Lie group used to describe the standard model of particle physics is a bit more complicated than what is above; it is $U(1) \times SU(2) \times SU(3)$ where $U(1)$ is again just complex phases, and $SU(N)$ is the group of all N by N unitary matrices with determinant one. Starting from this, one can go through the process of finding the corresponding Lie algebra, finding the operators which create massless spin-1 particles and then ultimately end up with the standard model of particle physics after also coupling the resulting gauge fields to the matter fields.

B. N=4 Super Yang-Mills Theories

This paper will be concerned with not just regular Yang-Mills theories but Yang-Mills theories with an extra symmetry added. This symmetry is called N=4 supersymmetry and comes from adding in extra scalar and spinor fields into our theory. This artificial symmetry has not been observed in the real world but provides us an opportunity to compute things that are extremely difficult to compute in the standard model and other quantum field theories. This means that we can learn a lot of lessons about the real world by working with this toy model.

Within the N=4 super Yang-Mills model, there exists what is called a moduli space of vacua. This is the space of all the possible ground states (minimum energy states) within an N=4 super Yang-Mills theory. It turns out that this space is actually continuous and in particular, it has a manifold-like structure. To give an example, if the underlying Lie group for the N=4 super Yang-Mills theory is $SU(N)$, then this moduli space of vacua (we'll call it M) is isomorphic to a $3(N-1)$ complex dimensional space modded out by a discrete group called the Weyl group, named after Hermann Weyl. This Weyl group is simply the group of permutations of N objects. Eqn. 4 shows this.

$$M \cong \mathbb{C}^{3(N-1)}/\Gamma \tag{3}$$

$$\Gamma \doteq Weyl(SU(N)) = S_N \tag{4}$$

Some general remarks about notation here are in order. When we write a space followed by a backslash and then a group, we mean that any two elements in the space which differ by the action of an element in the group are thought of as the same element. This is made clear in eqn. 5 and eqn. 6 where eqn. 5 implies eqn. 6.

$$G = F/\Gamma \tag{5}$$

$$G = \{[f] : f \in F\} \tag{6}$$
$$\text{where } [f_1] \equiv [f_2] \text{ iff } \exists g \in \Gamma \text{ s.t. } f_2 = gf_1$$

So, with this in mind, what eqn. 4 is telling us is that the moduli space of vacua is a $3(N-1)$ complex dimensional space where points in that space are identified with one another, if their coefficients are certain re-orderings of one another. This will be expanded upon in section 5. To make this more intuitive, let us consider the space $N = \mathbb{R}^2/\Gamma$. This space is regular euclidean 2-space where we have identified any points which are re-orderings of one another, so $(a, b) \equiv (b, a)$. Notice that the point (b, a) is just the point (a, b) flipped through the line $y = x$. This means N can be represented by just the portion of $\mathbb{R}^2$ that lies above (or equivalently, below) the line $y = x$.

II. Charge Lattices and Polarizations

As mentioned in the preceding section, in a Yang-Mills theory corresponding to a rank-N Lie algebra, there are N different massless spin-1 particles (”photons”) on the moduli space. Therefore, in a theory like this, every massive particle needs to have N different charges. To make this more concrete, consider a rank-2 Lie algebra, meaning we have 2 different massless spin-1 particles which can be thought of as two kinds of photons. So in such a theory a proton, for example, would have not one, but two electric charges. It would have a charge e_1 with respect to the first photon and a different charge e_2 with respect to the second photon. A particle also, in general, would have magnetic charges m_1 and m_2 with respect to each of the photons. So each massive state carries with it a vector $(\vec{e}, \vec{m})$ of electric and magnetic charges.

So we've learned that on the moduli space of a Yang-Mills theory, each massive particle has this vector of charges, but there is more to it. Given a system of two particles, the Dirac quantization condition gives us a constraint on what the charge vectors of the two particles can be. This quantization condition is stated in eqn. 7 where $(\vec{e}, \vec{m})_1$ are the vectors of electric and magnetic charge, respectively, for the first particle, and $(\vec{e}, \vec{m})_2$ are the vectors of electric and magnetic charge, respectively, for the second particle. The angle brackets represent some antisymmetric bilinear pairing. Thus the Dirac quantization condition requires that there exists some integral antisymmetric bilinear pairing between the electric and magnetic charges of two massive particles in a Yang-Mills theory.

$$\langle(\vec{e}, \vec{m})_1, (\vec{e}, \vec{m})_2\rangle \in \mathbb{Z} \tag{7}$$

After a bit of work, one can see that the Dirac quantization condition implies that the charge vector $(\vec{e}, \vec{m})$ of any particle is a member of a rank $2(N-1)$ lattice, where we are working in the context of an $SU(N)$ Yang-Mills theory with N=4 supersymmetry. The quantity $\langle(\vec{e}, \vec{m})_1, (\vec{e}, \vec{m})_2\rangle$ in the Dirac quantization condition is called the ”Dirac pairing” or ”polarization of the lattice of charges”. To be clear, a rank-2r lattice in our context is the set of points $\{z \in \mathbb{C}^r : z = a_1z_1 + a_2z_2 + a_3z_3 + ... + a_{2r}z_{2r}$, $a_i \in \mathbb{Z}$, $z_i \in \mathbb{C}^r$ are some fixed vectors that are linearly independent over the reals$\}$.

We say that the lattice is ”principally polarized” when the electric and magnetic charges span the entire $\mathbb{Z}^{2(N-1)}$ lattice (this is notation for a rank-$2(N-1)$ lattice) because when this is the case, it is possible to write the quantization condition as we have in eqn. 8 with the very simple looking matrix in the middle (a matrix with zeros along the diagonal, the identity matrix in the upper diagonal and minus the identity in the lower diagonal). We can imagine, however, that the charge vectors for a particle do not span the entire lattice, but rather span just a sublattice. In this case, we no longer have a principle polarization, and the matrix in eqn. 8 is no longer this simple.

$$(\vec{e}, \vec{m})_1 \begin{pmatrix} 0 & \mathbb{I}_{N-1} \\ -\mathbb{I}_{N-1} & 0 \end{pmatrix} \begin{pmatrix} \vec{e} \\ \vec{m} \end{pmatrix}_2 \tag{8}$$

In the case that we have a non-principle polarization of our charge lattice, there is a general theorem called ”the structure theorem for finitely generated modules over a principle ideal domain” which still gives us a general form for what the polarization will look like (this theorem is proven in chapter two, section six of [3]). This theorem tell us that there always exists a basis of the lattice where the polarization will look like what is shown in eqn. 9 where Δ is given in eqns. 10 and 11.

$$(\vec{e}, \vec{m})_1 \begin{pmatrix} 0 & \Delta \\ -\Delta & 0 \end{pmatrix} \begin{pmatrix} \vec{e} \\ \vec{m} \end{pmatrix}_2 \tag{9}$$

$$\Delta = diag\{\delta_1, \delta_2, ..., \delta_{N-1}\},\ \delta_i \in \mathbb{N} \tag{10}$$

$$\delta_i | \delta_{i+1}, i \in \{1, 2, ..., N-2\} \tag{11}$$

III. Abelian Varieties

An algebraic variety is simply a space which can be constructed as the zeros to a set of polynomial equations over either the real or complex numbers. If the space which the polynomial equations are written in happens to be a complex projective space (more on projective spaces can be read about in [5]) and the algebraic variety has a group structure, then the resulting variety is called an ”abelian variety”. This defintion turns out to be equivalent to defining an abelian variety as a complex torus subject to some constraints. I will now first show how to construct a complex torus, then I will explain what the constraints are such that this torus is also an abelian variety.

A N-dimensional complex torus can be constructed by taking N-dimensional complex space and modding it out by a rank $2N$ lattice. To illustrate this concept, I will take N to be one. Any higher N is very hard to visualize because we are already at a two complex dimensional (four real dimensional) space being modded out by a rank four lattice. Fig. 2 shows visually what it means to take a one complex dimensional space and mod it out by a rank two lattice. As mentioned before, the notation $\mathbb{C}/\Lambda$ means that any two points in $\mathbb{C}$ which differ by a point in the set $\Lambda = \{n_1\lambda_1 + n_2\lambda_2 : n_1, n_2 \in \mathbb{Z}\}$ are considered to be the same. The entire space $\mathbb{C}/\Lambda$ is then represented by equivalence classes where for any $z_1, z_2 \in \mathbb{C}$, $[z_1] \equiv [z_2]$ if and only if $z_1 - z_2 \in \Lambda$. So every equivalence class has a representative which lives inside of this parallelogram pictured on the left side of fig. 2. For this reason, we call the pictured parallelogram a ”unit cell” of Λ. Now, we see that each point along the left side of the unit cell is in the same equivalence class as the point straight across from it on the right side of the parallelogram (since these two points differ by the vector λ_1). So if we start at a point on the left hand side of the parallelogram and move straight across, when we hit the right side of the parallelogram, we are back at the same point. So we can think of this as moving around a circle. This is shown in the torus piece of fig. 2 where one loop around the center of the torus is labeled as λ_1. In a similar way, we can see why λ_2 is labelled by moving around the torus in the other independent direction. Another way to think about this action is by imagining you are holding a parallelogram shaped piece of paper in your hand. The left and right sides are identified,

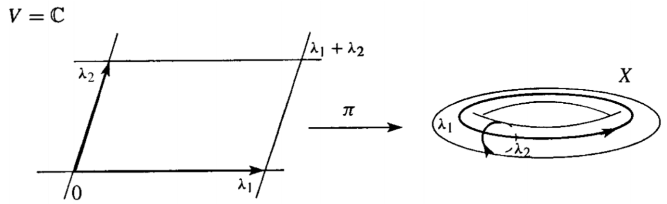
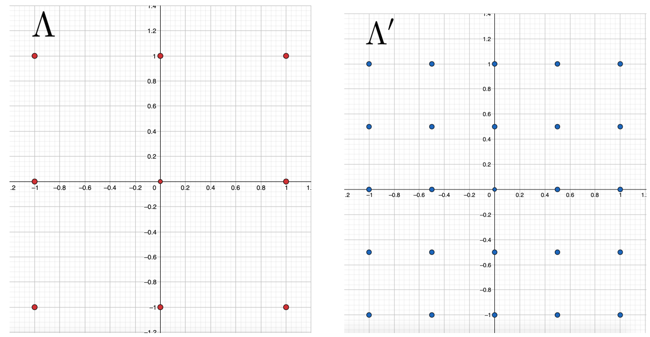


Fig. 2: This shows a one complex dimensional torus being constructed by taking the complex plane and modding it by a rank-2 lattice. λ_1 and λ_2 here are the basis vectors of the lattice Λ, X is the torus $\mathbb{C}/\Lambda$, and $\pi : \mathbb{C} \to X$ is the map taking us from the complex plane into the torus. This figure is taken from [5].

so you can represent this by bending the paper until the two sides meet and then gluing those sides together. Now you have a cylinder, but the caps of the cylinder are also identified. So now you bend your cylinder so that the two caps of it meet, and voilà, you have now created a torus.

It was mentioned above that in order for a complex torus to also be an abelian variety, some constraints must be met by the torus. These constraints have to do with the lattice Λ that mods out the complex space. In the $N = 1$ case, that is, when we take $\mathbb{C}$ modded by a rank-2 lattice, these constraints are simply that $\lambda_1 = 1$ and $\text{Im}(\lambda_2) > 0$. This may seem like a stringent condition, but no matter what λ_1, λ_2 we start with, we can always, by a holomorphic change of coordinates, move λ_1 to the point 1 and then the condition just turns out to be $\text{Im}(\lambda_2') > 0$ where λ_2' is whatever λ_2 ends up being after our holomorphic change of coordinates. A holomorphic change of coordinates is just a change of coordinates in which the function that does the action is holomorphic, i.e. if the function is called f, then $\frac{\partial f}{\partial \bar{z}} = 0$. This condition on f is completely equivalent to the Cauchy-Riemann equations which are a necessary and sufficient condition for a function f to be complex differentiable.

Now, to be more general than what is shown above, consider a torus $\mathbb{C}^N/\Lambda$ where Λ is now a rank-2N lattice with basis vectors $e_1 = (1, 0, 0, ..., 0)$, $e_2 = (0, 1, 0, ..., 0)$, ..., $e_N = (0, 0, 0, ..., 1)$, $e_{N+1} = (\tau^{11}, \tau^{12}, ..., \tau^{1N})$, $e_{N+2} = (\tau^{21}, \tau^{22}, ..., \tau^{2N})$, ..., $e_{2N} = (\tau^{N1}, \tau^{N2}, ..., \tau^{NN})$ where each $e_i \in \mathbb{C}^N$. It is always possible to write the basis in this way because, again, we can always use a holomorphic change of coordinates to move the first N basis vectors to vectors which have zeros in all but one component. If we consider the last N basis vectors to make up the matrix shown in eqn. 12 then the general condition which is necessary and sufficient for the torus $\mathbb{C}^N/\Lambda$ to also be an abelian variety is that $\det(\tau^{ij}) \neq 0$, $\tau^{ij} = \tau^{ji}$, and $\text{Im}(\tau^{ij}) > 0$ where the last condition here means that the matrix made up of just the imaginary part of each entry is a positive definite matrix (i.e. it has all positive eigenvalues). Proving the equivalence of an abelian variety defined as a projective algebraic variety and defined as a complex torus under the above constraints is a non-trivial exercise and is given in [6].



(a) An example lattice Λ.

(b) An example lattice Λ'.

Fig. 3: These lattices are used to visualize an isogeny between abelian varieties. Notice that $\Lambda \subset \Lambda'$.

$$\tau^{ij} = \begin{pmatrix} \tau^{11} & \tau^{12} & ... & \tau^{1N} \\ \tau^{21} & \tau^{22} & ... & \tau^{2N} \\ \vdots & \vdots & \vdots & \vdots \\ \tau^{N1} & \tau^{N2} & ... & \tau^{NN} \end{pmatrix} \tag{12}$$

IV. Isogenies Between Abelian Varieties

An isogeny is as a structure preserving map between algebraic groups (varieties that have a group structure) that is surjective and has a finite kernel. "Structure preserving" here has a different meaning depending on the specific context of the problem. When working in the context of linear algebra, a structure preserving map is just a linear transformation, but when working in group theory, a structure preserving map is a group homomorphism because this kind of a map preserves the notion of multiplication (i.e. $f(ab) = f(a)f(b)\ \forall a, b$). Applying the definition of an isogeny to abelian varieties defined as complex tori, we get that an isogeny between abelian varieties is a surjective map $h : X \to X'$ with a finite kernel where $X = \mathbb{C}^r/\Lambda$ and $X' = \mathbb{C}^r/\Lambda'$ with Λ and Λ' being rank-2r lattices with $\Lambda \subset \Lambda'$. The index, n, of an isogeny is defined to be $|ker(h)| = |\{x \in X : h(x) = 0\}|$.

To understand more intuitively what is meant by an isogeny, it is best to consider a simple example which is illustrated in fig. 3. Here we have two lattices Λ and Λ' embedded in the complex plane with Λ being a subset of Λ'. Let $(1) \in \mathbb{C}$ and $(i) \in \mathbb{C}$ be a basis of Λ and $(\frac{1}{2}) \in \mathbb{C}$ and $(\frac{i}{2}) \in \mathbb{C}$ be a basis of Λ'. Now consider X to be the torus $\mathbb{C}/\Lambda$ and X' to be the torus $\mathbb{C}/\Lambda'$. The isogeny $h : X \to X'$ then has kernel $ker(h) = \{[0], [\frac{1}{2}], [\frac{i}{2}], [\frac{1}{2} + \frac{i}{2}]\}$ where I have written these points in brackets to emphasize that the points in the torus are equivalence classes. We see that because $|ker(h)| = 4$, this particular isogeny has index 4. Another way people say this is by writing $[\Lambda' : \Lambda] = 4$. In general, we define $[\Lambda' : \Lambda] = |ker(h)|$ where Λ and Λ' have their usual interpretations as the lattices which mod out the complex plane, giving us our two tori. An equivalent way of thinking about the index of an isogeny between abelian varieties is by looking at the area covered by the unit cell of each torus. In our example above (fig. 3), the unit cell of Λ' has area $\frac{1}{4}$ while the unit cell of Λ has area 1. Thus the index of the isogeny is $\frac{1}{1/4} = 4$.

We now consider the following question: if given a lattice Λ', what are the possible sublattices of Λ', such that the sublattice induces an isogeny of index n? [7] gives us the answer. If we have a rank $2N$ lattice with basis given by $\{e_1, e_2, ..., e_{2N}\}$, then the possible inequivalent sublattices (that is, lattices $\Lambda \subset \Lambda'$) that induce an index-n isogeny have basis $\{b_1, b_2, ..., b_{2N}\}$ which is given in eqn. 13 with constraints given in eqn.14. The constraints tell us that the matrix in eqn. 13 has integers greater than zero along the diagonal, zeros above the diagonal and each entry below the diagonal must be between zero and whatever the corresponding diagonal entry is directly above it. The crucial extra constraint that is not shown in eqn. 14 is that *the product of the diagonal entries in this matrix is the index of the isogeny*.

$$\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{2N} \end{pmatrix} = \begin{pmatrix} n_1 & 0 & 0 & 0 & \dots \\ m_{1,1} & n_2 & 0 & 0 & \dots \\ m_{2,1} & m_{2,2} & n_3 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ m_{2N,1} & m_{2N,2} & m_{2N,3} & \dots & n_{2N} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_{2N} \end{pmatrix} \tag{13}$$

$$\begin{aligned} m_{i,j}, n_i &\in \mathbb{Z} \\ n_i &> 0 \\ 0 \le m_{i,j} < n_j \ \forall i,j &\in \mathbb{Z} \end{aligned} \tag{14}$$

This looks a bit complicated, so we will reduce it to a simple example to see what it is telling us. In the case that $N = 1$ and we are looking for index 2 isogenies, we have eqn. 15 where n_1 and n_2 are integers greater than zero such that $n_1 n_2 = 2$ and m is between zero and n_1.

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} n_1 & 0 \\ m & n_2 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \tag{15}$$

After a bit of algebra, it is not hard to see that this implies the only possible sublattices which induce an index-2 isogeny have bases given by $\{e_1, 2e_2\}$, $\{2e_1, e_2\}$, or $\{2e_1, e_1 + e_2\}$. So now we have built up some machinery in talking about abelian varieties and isogenies between them. We will now talk in depth about the structure of the moduli space of vacua in an SU(N) super Yang-Mills theory.

V. Moduli Space of Vacua for SU(N) Super Yang-Mills Theories

As mentioned in section 1.B, for a Yang-Mills theory with N=4 supersymmetry which is built from the SU(N) group, the moduli space of vacua is $\mathbb{C}^{3(N-1)}/\Gamma$ where Γ is the permutation group on N elements. This space can be simplified a bit, however. We will split the $3(N-1)$ copies of $\mathbb{C}$ into three copies of $\mathbb{C}^{N-1}$ because the group Γ acts on all of these independently and in equivalent ways. Additionally, each copy of $\mathbb{C}^{N-1}$ will be thought of as being embedded into C^N but with the constraint that the sum of all the components sum to zero. Thus our $\mathbb{C}^{N-1}$ is being thought of as an $N-1$ (complex) dimensional slice out of an N (complex) dimensional space. So now our problem is to look at what the space $\mathbb{C}^{N-1}/\Gamma$ looks like. To get a feel for this, we'll do two examples. First, we'll use the $SU(2)$ group to construct the Yang-Mills theory and then we'll use the $SU(3)$ group. We will see that in the $SU(2)$ case, our geometric picture of what's going on will be very helpful and will give us a nice intuition for what's going on. In the $SU(3)$ case, however, we will see how things very quickly become much more complicated and we must rely further on the algebra of the problem. After looking at these low dimensional examples, we will move on to the case of the more general SU(N) Lie group.

A. SU(2) Case

Here our moduli space of vacua looks like $\mathbb{C}^{3(2-1)}/\Gamma = \mathbb{C}^3/\Gamma$. But as we said before, each of the three copies of $\mathbb{C}$ here get acted on by Γ independently, so we can figure everything out by considering $\mathbb{C}/\Gamma$ and then at the end remembering that the full result is whatever we get "times three". So, to begin recall that we want to think of $\mathbb{C}$ as being a slice of $\mathbb{C}^2$. So the coordinates of our moduli space will be labeled as $(z_1, z_2) \in \mathbb{C}^2$ under the constraint that $z_1 + z_2 = 0 \implies z_1 = -z_2$. The group Γ here is the permutation group on two elements. It is easy to see that this group itself has just two elements, namely, the identity element that takes (z_1, z_2) to (z_1, z_2), and the element that swaps the two coordinates. Thus Γ has only one generator and it is the elements $g_1 : (z_1, z_2) \to (z_2, z_1)$.

In general, when we take a space with a manifold structure and mod it out by a group, we get something called an orbifold. Roughly, this is a space that is smooth at a generic point, but has singularities at others. The singularities are at precisely the points which are fixed by the group action. With this in mind, let's figure out which points in $\mathbb{C}^2$ are fixed by Γ. If a point z is fixed by Γ, then that means that there exists some $g \in \Gamma$ which is not the identity element such that $g(z) = z$. In this context, this means that there is some (z_1, z_2) such that $g_1((z_1, z_2)) = (z_1, z_2)$. But $g_1((z_1, z_2)) = (z_2, z_1)$. So this implies that $(z_2, z_1) = (z_1, z_2)$. Thus, the points which are fixed are the points (z_1, z_2) such that $z_1 = z_2$. But recall that we have the constraint $z_1 + z_2 = 0$. So this means the space $\mathbb{C}/\Gamma$ has just one singularity, and it is the point $(0, 0)$.

Now looking at the space $\mathbb{C}/\Gamma$ more generally, notice that Γ is isomorphic to the group $\mathbb{Z}_2$. A generator of the group $\mathbb{Z}_2$ is the map $g_2 : e^{i\theta} \to e^{i(\theta+\pi)}$. We can think of it in this way because if we apply g_2 twice, we get back to the original point. So $\mathbb{C}/\Gamma$ is the complex plane where points which differ by an angle of π radians are identified. To visualize this space, imagine taking a sheet of paper (this is the complex plane), cutting a slit in it along the minus-x axis, and then folding the paper into itself so that points which were originally separated by a 180 degree rotation are now on top of one another. Now you have a cone, and this is exactly the geometric interpretation of the moduli space of vacua of the SU(2) super Yang-Mills theory that we sought out. In addition, we recognize that the origin is clearly a singularity here since the curvature become infinite (it is the tip of the cone). Remember, however, that this is just one of three copies of the complex plane that we originally started with. The total moduli space has complex dimension three and is thus already extremely difficult to picture.

B. SU(3) Case

Now we will consider the moduli space of vacua of the super Yang-Mills theory arising from the underlying group $SU(3)$. Here, our moduli space of vacua looks like $\mathbb{C}^{3(3-1)}/\Gamma = \mathbb{C}^6/\Gamma$. Once again, however, we will just begin by looking at $\mathbb{C}^2/\Gamma$ where Γ is now the permutation group acting on three objects and we think of $\mathbb{C}^2$ as being embedded in $\mathbb{C}^3$ subject to the constraint $z_1+z_2+z_3=0$ where z_1,z_2,z_3 are the coordinates in $\mathbb{C}^3$. There are two generators to the group of permutations on three objects. They can be taken to be a few different specific members of the group, but I will take them to be the elements $g_3 : (z_1,z_2,z_3) \to (z_2,z_1,z_3)$ and $g_4 : (z_1,z_2,z_3) \to (z_3,z_1,z_2)$. There are six total elements in Γ which can all be built out of g_3 and g_4. Other than the identity, g_3, and g_4, the remaining elements of Γ are $g_4^{-1}g_3g_4 : (z_1,z_2,z_3) \to (z_3,z_2,z_1)$, $g_4g_3g_4^{-1} : (z_1,z_2,z_3) \to (z_1,z_3,z_2)$, and $g_4^2 : (z_1,z_2,z_3) \to (z_2,z_3,z_1)$. Let's find out what the fixed points are of each of these elements.

I will simply state the results of finding the fixed points of each of these elements but it is easy to check that it is true just by applying the group element to the corresponding fixed point. The fixed points of g_3 are any points with $z_1=z_2$. The fixed points of g_4 are any points which have $z_1=z_2=z_3$, but remembering that we require $z_1+z_2+z_3=0$, this means that $z_1=z_2=z_3=0$. The fixed points of $g_4^{-1}g_3g_4$ are the points with $z_2=z_3$, and finally the fixed points of $g_4g_3g_4^{-1}$ are the points with $z_1=z_3$. Notice also that the fixed points of g_4^2 are exactly the fixed points of g_4, that is, $z_1=z_2=z_3=0$. So now we impose the condition that the sum of the components equals zero to get rid of, say, z_3. The result of this is that our space of fixed points, that is, the space of the singularities of our space of vacua, are the following sets of points: $(0,0,0), \{(z_1,z_2) : z_1 = z_2\}, \{(z_1,z_2) : z_1 = -2z_2\}, \{(z_1,z_2) : z_1 = -\frac{1}{2}z_2\}$. This has been shown pictorially in fig. 4. This figure needs to be interpreted correctly though. The entire space here is $\mathbb{C}^2$, which is actually a four (real) dimensional space. So these points of singularities which look like lines in the picture are actually two (real) dimensional planes of singular space, analogous to the tip of the cone that we saw in the example with the SU(2) Lie group.

There is one further complication. Recall that the lines pictured in fig. 4 are the lines $z_1=z_2$, $z_2=z_3$, and $z_1=z_3$. But g_3 cyclically permutes the coordinates, so it takes the equation $z_1=z_2$ to the equation $z_2=z_3$. Similarly, g_3 takes the equation $z_2=z_3$ to $z_3=z_1$ and it takes $z_3=z_1$ to $z_1=z_2$. So all of these equations are related to one another by g_3 and thus are all actually the same line in our moduli space because our moduli space is constructed by taking $\mathbb{C}^2$ and modding it out by a group which contains g_3. So these three

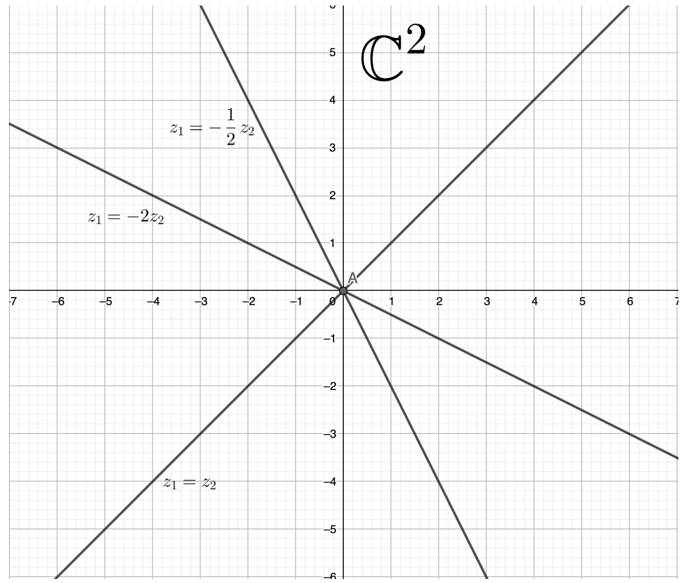


Fig. 4: This shows the singularities of $\mathbb{C}^2/\Gamma$ which were found by computing the fixed points of the group Γ. The horizontal axis is meant to represent one complex variable z_1 and the vertical axis represents another complex variable z_2.

lines all collapse onto one another when the space is modded out by Γ. The total space is what is pictured in fig. 4 but folded around itself in a way analogous to how we constructed the cone in the SU(2) case. The folds are just right so that all the lines in the figure fall onto one another. The resulting total moduli space looks like the four (real) dimensional analogue of a cone with a tip corresponding to the origin and an entire two (real) dimensional linear subspace of the cone being singular. This is shown in fig. 5 where the orange line is supposed to represent the two (real) dimensional singular subspace. The point on top is also singular. Remember, though, that this is just one-third of the story. The total moduli space looks like three copies of what was described above. This illustrates the purpose of my statement previously that these spaces of vacua become complicated quite quickly and that we should rely on algebra rather than geometry to tell us what's going on. It's essentially impossible to picture this space in a perfect way where you could make meaningful predictions from the picture because in order to do so, you would need to be picturing a quite complicated four (real) dimensional space and even then you're only talking about a four (real) dimensional slice of a twelve (real) dimensional space.

C. SU(N) Case

Now we move on to the general case of an SU(N) super Yang-Mills theory. It will be helpful to keep the above low dimensional examples in mind when talking about this general case so that one can have some sort of intuition for what's going on. As a reminder, once again the moduli space of vacua here looks like $M = \mathbb{C}^{3(N-1)}/\Gamma$ where Γ is the permutation group on N elements. At the fixed points of Γ, just as we saw in the previous example, we get singularities. These

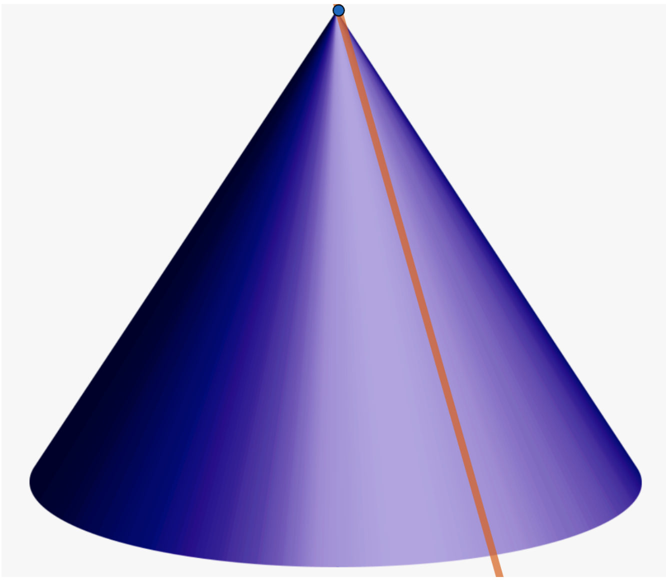


Fig. 5: Shown here is a cartoon version of what the moduli space of vacua for an SU(3) Yang-Mills theory is. The actual space is the four dimensional analogue of a cone. The orange slice represents a 2 dimensional subspace which is singular.

singularities could be individual points but they could also be lines, planes, or even hyper-planes in higher dimensions. At a generic point, that is, a non-fixed point in $\mathbb{C}^{N-1}$ (because remember Γ act independently on the three copies of $\mathbb{C}^{N-1}$ that make up the moduli space), we just have flat $N-1$ complex dimensional space but maybe folded into a hyper-cone like structure. At these generic points, the operators of the Yang-Mills theory which create massless spin-1 particles are exactly the same operators as in an N=4 super $U(1)^{N-1}$ Yang-Mills theory. $U(1)$ is the guage group of quantum electrodynamics, and the massless spin-1 fields of this theory are just photon fields. So, on a generic point of our moduli space of vacua, our theory looks *exactly* like $N-1$ copies of electromagnetism. That is, we have what was described in the introduction as a situation where there are $N-1$ types of photon analogues meaning every massive particle carries $N-1$ charges.

On a non-generic point of the space of vacua, we have nested singular subspaces. The gauge groups of these singular subspaces are nested as well in the following way for an $SU(N)$ Yang-Mills theory. The generic point, as mentioned before, has a gauge group $U(1)^{N-1}$ and is $3(N-1)$ (complex) dimensional. Within the total space, there is a singular subspace which has gauge group $SU(2)\times U(1)^{N-2}$ and has (complex) dimension $3(N-2)$. Within this singular subspace, there is another singular subspace which has gauge group $SU(3)\times U(1)^{N-3}$ and has (complex) dimension $3(N-3)$. This continues on until we finally get to the case where we have a three (complex) dimensional subspace with gauge group $SU(N-1)\times U(1)$.

VI. The Question at Hand

So we know what the moduli space of vacua looks like for an $SU(N)$ super Yang-Mills theory. We are now in a position to present the question which is being explored through our current research. We want to understand what happens to the polarization of charge lattices in the space of vacua as we approach the singular subspaces. In particular, we want to learn about the charge lattices of $SU(M)$ (with $M<N$) sub-theories living in the singularities of the moduli space of the $SU(N)$ theories. These charge lattices will have rank $2(M-1)$ and we want to understand how this rank-$2(M-1)$ lattice will be embedded in the larger rank-$2(N-1)$ lattice which comes from the total $SU(N)$ theory. The physics of our situation tells us that this sublattice will induce an isogeny between the corresponding abelian varieties associated to the charge lattices. Then finally, our quest is to find out what the δ_i factors from eqn. 10 are for the polarization of the lattice sitting inside the singular subspace.

Understanding this one question involves understanding the long story of the moduli space of vacua of Yang-Mills theories in addition to some specialized math topics, but I hope that I have laid out in a clear way all the necessary background for this question. The main picture to keep in mind is that we are working in this moduli space of vacua (which has a very weird shape) and every point in this space has a charge lattice associated with it. This lattice itself sits inside of a complex space, so instead of thinking of the charge lattices, we can think of the complex space mod the charge lattices, which are called abelian varieties (tori). So each point in the moduli space has a torus associated with it and we want to understand what happens to the polarization of the torus as we approach (in the moduli space) the singular subspaces of the moduli space. Approaching these singular subspaces corresponds to taking isogenies between the abelian varieties.

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